

Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory

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Abstract

We argue that the scattering amplitudes in the maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills theory possess a new symmetry which extends the previously discovered dual conformal symmetry. To reveal this property we formulate the scattering amplitudes as functions in the appropriate dual superspace. Rewritten in this form, all tree-level MHV and next-to-MHV amplitudes exhibit manifest dual superconformal symmetry. We propose a new, compact and Lorentz covariant formula for the tree-level NMHV amplitudes for arbitrary numbers and types of external particles. The dual conformal symmetry is broken at loop level by infrared divergences. However, we provide evidence that the anomalous contribution to the MHV and NMHV superamplitudes is the same and, therefore, their ratio is a dual conformal invariant function. We identify this function by an explicit calculation of the six-particle amplitudes at one loop. We conjecture that these properties hold for all, MHV and non-MHV, superamplitudes in $\mathcal{N} = 4$ SYM both at weak and at strong coupling.

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1 Introduction

The scattering amplitudes in maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) have a number of remarkable properties both at weak and at strong coupling. Defined as matrix elements of the S -matrix between asymptotic on-shell states, they inherit the symmetries of the underlying gauge theory. In addition, trying to understand the properties of the scattering amplitudes, one can discover new dynamical symmetries of the $\mathcal{N} = 4$ theory. A well-known example is the twistor formulation of the scattering amplitudes by Witten [1].

In this paper we argue that the planar scattering amplitudes in $\mathcal{N} = 4$ SYM theory have a hidden symmetry that we call *dual superconformal symmetry*. It appears on top of all well-known symmetries of the scattering amplitudes (supersymmetry, conformal symmetry, etc.) and is not related (not in an obvious way, at least) to an invariance of the Lagrangian of the theory. Quite remarkably, the same symmetry also emerges, in the planar limit, from fermionic T-duality of the sigma model on an $\text{AdS}_5 \times \text{S}^5$ background in the AdS/CFT description of scattering amplitudes [2].

Hints of such a symmetry first came from the classification of the loop integrals entering the perturbative expansion of the planar four-gluon maximally helicity violating (MHV) scattering amplitude [3]. The latter was calculated up to three loops in terms of a restricted class of planar integrals by Bern, Dixon and Smirnov (BDS) in [4]. In [5] it was observed that these integrals all have a very special property. If one changes variables from momenta p_i^μ to ‘dual coordinates’ x_i^μ via

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu, \quad (1.1)$$

then the integrals exhibit a formal conformal covariance in the dual x -space. This dual conformal symmetry is formal because the integrals are in fact infrared divergent and the introduction of dimensional regularisation breaks the symmetry. Nevertheless, even broken, the dual conformal symmetry still imposes constraints on the on-shell scattering amplitudes. Their precise formulation required further developments, both at strong and weak coupling. At strong coupling, Alday and Maldacena applied the AdS/CFT correspondence to the study of scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills [6]. According to their proposal, the planar n -gluon amplitudes with various helicity configurations are related at strong coupling to the area of a minimal surface in AdS_5 space. This surface is attached to a closed polygon contour C_n made of n light-like segments $[x_i, x_{i+1}]$, defined by the on-shell gluon momenta according to (1.1). In the case of the four-particle amplitude the area could be calculated explicitly and was shown to agree with the ABDK/BDS conjecture about the all-order structure of MHV amplitudes [3, 4]. An important observation made in [6] was that the calculation of the amplitude is mathematically identical to that of a Wilson loop $W(C_n)$ at strong coupling [7].

These findings together with previous results on the relation between infrared singularities of scattering amplitudes and Wilson loops [8] subsequently lead to a formulation of the duality between scattering amplitudes and light-like Wilson loops at weak coupling (see Fig. 1). Firstly, in [9] it was found that at one-loop the four-point MHV amplitude matched the corresponding Wilson loop $W(C_4)$. This was then generalised to n points at one loop [10]. The duality was shown to hold beyond one loop at four [11], five [12] and six [13, 14] points. As discussed in [15], the confirmation of the duality at six points and two loops necessarily implies the breakdown of the BDS conjecture beyond five points. Indeed, evidence that the conjecture should fail for some number of gluons had been found in [16]. The analysis of the Regge limits also shows the BDS conjecture had indeed to fail at six points and two loops [17]. This was later confirmed by an explicit two-loop six-gluon calculation [14]. Not only was it shown that the BDS conjecture fails

at two loops, but it fails exactly so that the duality with the Wilson loop is maintained [13, 14].

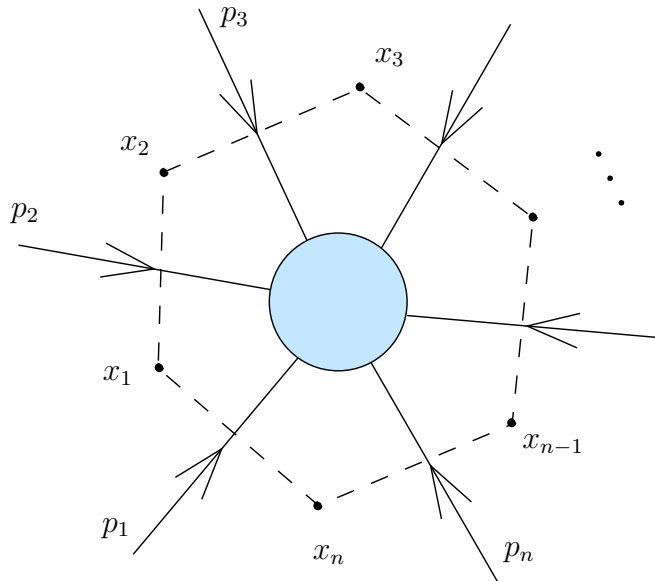


Figure 1: Diagrammatic representation of the scattering amplitude/Wilson loop duality. The dashed line depicts the contour C_n .

An important feature of this duality is that the Wilson loop in $\mathcal{N} = 4$ SYM has a natural conformal symmetry. Applied to the dual scattering amplitudes, this implies an unexpected ‘dual’ conformal symmetry, acting on the particle momenta via their relation to the dual coordinates x_i (1.1). The conformal symmetry of the Wilson loop is formulated as an anomalous conformal Ward identity (the anomaly is due to the ultraviolet divergences of the Wilson loop) [11, 12]. This symmetry is sufficient to fix the form of the finite part of the four- and five-cusp Wilson loops, both at weak [11, 12] and at strong coupling [16, 18]. For six-cusp Wilson loops and beyond, the conformal symmetry leaves some freedom. So, the fact that the MHV amplitude continues to agree with the Wilson loop at this level shows that there is more to the amplitude/Wilson loop duality than just dual conformal symmetry.

The above developments referred to MHV amplitudes. A remarkably simple features of MHV amplitudes is the fact that supersymmetry allows to reduce the problem of calculating all n -particle MHV amplitudes to computing a single scalar function of the Mandelstam variables. Amplitudes with other helicity configurations, such as next-to-MHV (NMHV), next-to-next-to-MHV (N^2 MHV) and so on, are known to have a more complicated structure [19, 20, 21, 22, 23]. In this paper we provide evidence that amplitudes with arbitrary helicity configurations enjoy a new, bigger symmetry which can be thought of as a supersymmetric generalization of dual conformal symmetry.

In the planar limit, a generic n -particle scattering amplitude in the $\mathcal{N} = 4$ SYM theory with an $SU(N)$ gauge group has the form

$$\mathcal{A}_n(\{p_i, h_i, a_i\}) = (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n p_i\right) \sum_{\sigma \in S_n/Z_n} 2^{n/2} g^{n-2} \text{tr}[t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n)}}] A_n(\sigma(1^{h_1}, \dots, n^{h_n})) , \quad (1.2)$$

where each scattered particle (scalar, gluino with helicity $\pm 1/2$ or gluon with helicity ± 1) is characterized by its on-shell momentum p_i^μ ($p_i^2 = 0$), helicity h_i and color index a_i . Here the sum

runs over all possible non-cyclic permutations σ of the set $\{1, \dots, n\}$ and the color trace involves the generators t^a of $SU(N)$ in the fundamental representation normalized as $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. All particles are treated as incoming, so that the momentum conservation takes the form $\sum_{i=1}^n p_i = 0$.

The color-ordered partial amplitudes $A_n(\sigma(1^{h_1}, \dots, n^{h_n}))$ only depend on the momenta and helicities of the particles and admit a perturbative expansion in powers of 't Hooft coupling $a = g^2 N / (8\pi^2)$. The best studied so far are the gluon scattering amplitudes. In some cases, amplitudes with external particles other than gluons can be obtained from them with the help of supersymmetry. In particular, the supersymmetric Ward identities [24] imply that the partial amplitudes A_n vanish to all orders in the coupling when at least $n - 1$ gluons have the same helicity,¹

$$A_n(1^\pm, 2^+, \dots, n^+) = 0. \quad (1.3)$$

In the same way, for maximally helicity violating amplitudes (MHV) amplitudes, i.e. amplitudes with two gluons of negative helicity, say the j -th and the k -th, all perturbative corrections can be factorized into a universal scalar factor $M_n^{(\text{MHV})}$ independent of j and k

$$A_n^{\text{MHV}}(1^+ \dots j^- \dots k^- \dots n^+) = A_{n;0}^{\text{MHV}} + a A_{n;1}^{\text{MHV}} + O(a^2) = A_{n;0}^{\text{MHV}} M_n^{\text{MHV}}. \quad (1.4)$$

Here $M_n^{\text{MHV}} = M_n^{\text{MHV}}(\{s_{ij}\}, a)$ is a function of the Mandelstam kinematical invariants and of the 't Hooft coupling. The tree amplitude is given, in the spinor helicity formalism $p^{\dot{\alpha}\alpha} = (\sigma_\mu)^{\dot{\alpha}\alpha} p^\mu = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$, by the Parke-Taylor formula [25, 26]

$$A_{n;0}^{\text{MHV}} = i \frac{\langle j k \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad (1.5)$$

where $\langle jk \rangle = \epsilon_{\alpha\beta} \lambda_j^\alpha \lambda_k^\beta$ and λ_j^α (or $\tilde{\lambda}_j^{\dot{\alpha}}$) is a two-component Weyl commuting spinor with helicity $-1/2$ (or $+1/2$). Two special features of the MHV amplitude (1.4) are that, firstly, the tree-level factor is holomorphic in the λ -spinors and, secondly, the same function of spinors (1.5) appears to all orders in 't Hooft coupling. This allows one to reduce the problem of calculating the MHV amplitude to all loops to determining the scalar function M_n^{MHV} . We shall return to this function shortly.

The above-mentioned simple properties of the MHV amplitudes are already lost for next-to-MHV gluon amplitudes A_n (with $n \geq 6$) which have three gluons of negative helicity. In that case, the tree-level amplitude $A_{n;0}^{\text{NMHV}}(\lambda, \tilde{\lambda})$ depends on both λ - and $\tilde{\lambda}$ -spinors and, most importantly, the perturbative corrections produce new helicity structures $A_{n;1}^{\text{NMHV},(\ell)}(\lambda, \tilde{\lambda})$ different from the tree-level one [19, 20, 21, 22, 23],

$$A_n^{\text{NMHV}}(1^+ \dots i^- \dots j^- \dots k^- \dots n^+) = A_{n;0}^{\text{NMHV}}(\lambda, \tilde{\lambda}) + a \sum_{\ell} A_{n;1}^{\text{NMHV},(\ell)}(\lambda, \tilde{\lambda}) M_{n;1}^{\text{NMHV},(\ell)} + O(a^2). \quad (1.6)$$

Another complication with the NMHV amplitude is due to the fact that the coefficients $A_{n;1}^{\text{NMHV},(\ell)}$ and the corresponding scalar Feynman integrals $M_{n;1}^{\text{NMHV},(\ell)}$ also depend on the positions of the gluons i, j and k carrying negative helicities and, therefore, they have to be calculated separately for each configuration of helicities. A general expression for the one-loop NMHV gluon amplitude was given in [21] and it was later generalized to amplitudes with external particles different from

¹This is true for $n \geq 4$. However, for $n = 3$ one can construct e.g. amplitudes $A_3(1^-, 2^+, 3^+) \neq 0$ provided the on-shell momenta are complex [1].

gluons in [27]. The situation becomes even more complex for the $N^2\text{MHV}$, \dots amplitudes. In this case only certain parts of the one-loop amplitudes have been constructed using the quadruple cut method [22], and at tree level only the restricted class of the so-called split-helicity amplitudes were found [28].

Bearing in mind the simplicity of the Parke-Taylor formula (1.5) for MHV amplitudes, one may wonder whether the complexity of the NMHV amplitudes is genuine or whether there exists some symmetry which relates different NMHV amplitudes to each other and which allows us to write them all in a compact form. In this paper we show that such a symmetry does exist. As a hint, we return to the MHV amplitudes and recall [29] that the various tree-level MHV amplitudes can be combined into a single MHV superamplitude by introducing Grassmann variables η_i^A (with $A = 1, \dots, 4$), one for each external particle. Since the perturbative corrections to the MHV amplitude are factorized into a universal scalar factor (1.4), we can write the all-loop generalization of the MHV superamplitude as

$$\mathcal{A}_n^{\text{MHV}}(p_1, \eta_1; \dots; p_n, \eta_n) = i(2\pi)^4 \frac{\delta^{(4)}\left(\sum_{j=1}^n p_j\right) \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} M_n^{\text{MHV}}, \quad (1.7)$$

where $\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right) = \prod_{\alpha=1,2} \prod_{A=1}^4 \lambda_{i\alpha} \eta_i^A$ is a Grassmann delta function. The all-loop MHV amplitudes appear as coefficients in the expansion of $\mathcal{A}_n^{\text{MHV}}$ in powers of η_i in such a way that $(\eta_i)^h \equiv \prod_{k=1}^h \eta_i^{A_k}$ is associated with an external particle with momentum p_i and helicity $1 - h/2$. In particular, the gluon MHV amplitude (1.4) arises as

$$\mathcal{A}_n^{\text{MHV}} = (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n p_i\right) \sum_{1 \leq j < k \leq n} (\eta_j)^4 (\eta_k)^4 A_n^{\text{MHV}}(1^+ \dots j^- \dots k^- \dots n^+) + \dots, \quad (1.8)$$

where the dots denote terms describing MHV amplitudes with some external particles different from gluons. As was already mentioned, the function M_n^{MHV} exhibits a very interesting symmetry when expressed in terms of the dual coordinates defined in (1.1). Explicit perturbative calculations show [3, 4, 30] that M_n^{MHV} is given by a sum of Feynman integrals which are formally (in $D = 4$ dimensions) covariant under the $SO(2, 4)$ transformations of the dual coordinates x_i^μ . Moreover, the conjectured duality between the MHV scattering amplitudes and light-like Wilson loops leads to the following relation [9, 10],

$$\ln M_n^{\text{MHV}} = \ln W_n + \text{const} + O(\epsilon, 1/N^2), \quad (1.9)$$

where $W_n = W(C_n)$ is the expectation value of a Wilson loop evaluated over the light-like polygonal contour C_n in Minkowski space-time, with vertices located at the points x_i^μ (with $i = 1, \dots, n$),

$$W_n = \frac{1}{N} \langle 0 | \text{tr} P \exp \left(ig \oint_{C_n} dx^\mu A_\mu \right) | 0 \rangle. \quad (1.10)$$

The divergences of M_n^{MHV} and W_n in (1.9) are regularized using dimensional regularization with $D = 4 - 2\epsilon$. As mentioned earlier, the natural (anomalous) conformal symmetry of the Wilson loop induces a ‘dual’ conformal symmetry of the MHV amplitude.

In the present paper we show that the MHV superamplitude (1.7) possesses an even bigger, dual superconformal symmetry. This symmetry acts on the dual coordinates x_i^μ and their

superpartners $\theta_{i\alpha}^A$ which are defined in close analogy with (1.1) in terms of odd variables η as follows,

$$\lambda_i^\alpha \eta_i^A = \theta_i^{A\alpha} - \theta_{i+1}^{A\alpha}. \quad (1.11)$$

We demonstrate that when expressed in terms of the dual supercoordinates $(x_i, \lambda_i^\alpha, \theta_i^{A\alpha})$ the MHV superamplitude (1.7) transforms covariantly under $\mathcal{N} = 4$ superconformal ($SU(2, 2|4)$) transformations. Most importantly, dual superconformal symmetry also allows us to understand much better the complicated structure of the one-loop NMHV amplitudes. We argue that, in a close analogy with the MHV amplitudes, all NMHV amplitudes can be combined into a single superamplitude $\mathcal{A}_n^{\text{NMHV}}$ which is a homogeneous polynomial of degree 12 in the Grassmann variables η_i^A . Similarly to (1.8), e.g., the gluon NMHV amplitudes arise as the coefficients in front of $(\eta_i)^4(\eta_j)^4(\eta_k)^4$,

$$\mathcal{A}_n^{\text{NMHV}} = (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^n p_i \right) \sum_{i,j,k} (\eta_i)^4 (\eta_j)^4 (\eta_k)^4 A_n^{\text{NMHV}} (1^+ \dots i^- \dots j^- \dots k^- \dots n^+) + \dots \quad (1.12)$$

Quite remarkably, $\mathcal{A}_n^{\text{NMHV}}$ transforms covariantly under the dual superconformal transformations and has the same conformal weights as the MHV superamplitude $\mathcal{A}_n^{\text{MHV}}$. As a result, the ‘ratio’ of the two superamplitudes is given by a linear combination of *superinvariants* of the form (to one-loop order)

$$\mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \times \left(\frac{1}{n} \sum_{p,q,r=1}^n c_{pqr} \delta^{(4)}(\Xi_{pqr}) [1 + a V_{pqr} + O(\epsilon)] + O(a^2) \right). \quad (1.13)$$

Here $\delta^{(4)}(\Xi_{pqr}) \equiv \prod_{A=1}^4 \Xi_{pqr}^A$ are Grassmann delta functions. The integers $p \neq q \neq r$ label three points in the dual superspace $(x_i, \lambda_i^\alpha, \theta_i^{A\alpha})$. From the coordinates of these three points one makes the dual superconformal covariants $\Xi_{pqr}(x, \lambda, \theta)$, linear in the odd variables θ . Both Ξ_{pqr} and the c-valued coefficients $c_{pqr}(x, \lambda)$ transform covariantly under dual superconformal transformations in such a way that the product $c_{pqr} \delta^{(4)}(\Xi_{pqr})$ remains invariant. Equivalently, the dual superconformal invariants can be rewritten as functions of the on-shell superspace coordinates $(\lambda_i, \tilde{\lambda}_i, \eta_i)$ as follows,

$$c_{pqr} = c_{pqr}(\lambda, \tilde{\lambda}), \quad \Xi_{pqr} = \Xi_{pqr}(\lambda, \tilde{\lambda}, \eta), \quad (1.14)$$

with the corresponding induced action of the dual superconformal algebra, in accord with the defining relations (1.1) and (1.11).

The dependence on the coupling constant enters the right-hand side of (1.13) through the perturbative corrections to the MHV superamplitude (1.4) and through the scalar factor involving the functions $V_{pqr}(x_i)$. According to the duality relation (1.9), the former are determined by the light-like Wilson loop W_n . Unlike W_n , the functions $V_{pqr}(x_i)$ remain finite as $\epsilon \rightarrow 0$ and, most importantly, they are exactly invariant under dual conformal transformations of x_i^μ . This means that the dual conformal symmetry of the superamplitudes $\mathcal{A}_n^{\text{MHV}}$ and $\mathcal{A}_n^{\text{NMHV}}$ is broken by the infrared divergences in such a way that the factor in parentheses in the right-hand side of (1.13), to which we shall refer as the ‘ratio’ of the two superamplitudes, remains dual conformal as $\epsilon \rightarrow 0$.

We demonstrate this property by an explicit one-loop calculation for $n = 6$ and we conjecture

that it should be true to all loops and for *all* superamplitudes in the $\mathcal{N} = 4$ SYM theory,

$$\begin{aligned}\mathcal{A}_n &\equiv \mathcal{A}_n^{\text{MHV}} + \mathcal{A}_n^{\text{NMHV}} + \mathcal{A}_n^{\text{N}^2\text{MHV}} + \dots + \mathcal{A}_n^{\text{N}^{n-4}\text{MHV}} \\ &= \mathcal{A}_n^{\text{MHV}} \left[R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) + O(\epsilon) \right].\end{aligned}\quad (1.15)$$

In the first of these relations the sum runs over all N...NMHV amplitudes and $\mathcal{A}_n^{\text{N}^{n-4}\text{MHV}} = \mathcal{A}_n^{\overline{\text{MHV}}}$ is the googly, or anti-MHV amplitude. The ratio function

$$R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) = 1 + R_n^{\text{NMHV}} + R_n^{\text{N}^2\text{MHV}} + \dots + R_n^{\text{N}^{n-4}\text{MHV}} \quad (1.16)$$

is finite as $\epsilon \rightarrow 0$. Most importantly, it satisfies the dual conformal Ward identities

$$K^\mu R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) = D R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) = 0, \quad (1.17)$$

with the dilatation D and conformal boost K^μ operators defined appropriately in the on-shell superspace $(\eta_i, \lambda_i, \tilde{\lambda}_i)$.

In the simplest case $n = 6$, the general expression for the NMHV superamplitude (1.13) simplifies due to the fact that all possible supercovariants Ξ_{pqr} can be expressed in terms of Ξ_{146} and five other supercovariants obtained from Ξ_{146} by consecutive cyclic shift of the indices $i \rightarrow i + 1$, with the periodicity condition $i + 6 \equiv i$. This leads to the following remarkably simple one-loop expression for the $n = 6$ NMHV superamplitude

$$\mathcal{A}_6^{\text{NMHV}} = \mathcal{A}_6^{\text{MHV}} \left[\frac{1}{2} c_{146} \delta^{(4)}(\Xi_{146}) (1 + a V_{146}) + (\text{cyclic}) + O(a^2) \right], \quad (1.18)$$

where the MHV superamplitude $\mathcal{A}_6^{\text{MHV}}$ is given by (1.7) and (1.9). Here Ξ_{146} can be expressed in terms of three Grassmann η -variables

$$\Xi_{146} = \langle 61 \rangle \langle 45 \rangle (\eta_4 [56] + \eta_5 [64] + \eta_6 [45]), \quad (1.19)$$

while the function c_{146} is given by

$$c_{146} = - \frac{\langle 34 \rangle \langle 56 \rangle}{x_{46}^2 \langle 1 | x_{16} x_{63} | 3 \rangle \langle 1 | x_{16} x_{64} | 4 \rangle \langle 1 | x_{14} x_{45} | 5 \rangle \langle 1 | x_{14} x_{46} | 6 \rangle}, \quad (1.20)$$

where the standard notation for contracting spinors $\lambda, \tilde{\lambda}$ and vectors x is used (see Appendix A for more details). Finally, the scalar function V_{146} is given by a linear combination of scalar (1-mass, 2-mass-hard and 2-mass-easy in the nomenclature of [31]) box integrals

$$V_{146} = -\ln u_1 \ln u_2 + \frac{1}{2} \sum_{k=1}^3 \left[\ln u_k \ln u_{k+1} + \text{Li}_2(1 - u_k) \right], \quad (1.21)$$

where the periodicity condition $u_{i+3} = u_i$ is implied. This function depends on the conformally invariant cross-ratios u_1, u_2 and u_3 made from the dual coordinates,

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}. \quad (1.22)$$

Expanding the right-hand side of the relation (1.18) in powers of η 's and comparing the result with (1.12), it is straightforward to extract the expressions for the three different six-gluon NMHV one-loop amplitudes A^{+++---} , A^{++-+--} and A^{+-+--+} and to verify that they agree with the known results [19]. In the same manner, one can also verify that the superamplitude (1.18) correctly reproduces the one-loop expressions for various NMHV amplitudes involving scalars and gluinos [27].

The paper² is organized as follows. In Sect. 2 we discuss the dual conformal properties of gluon amplitudes. After recalling the formulation of MHV amplitudes in terms of commuting spinors λ_α and $\tilde{\lambda}_{\dot{\alpha}}$, we introduce dual coordinates $x_{\alpha\dot{\alpha}}$ and the notions of on-shell (coordinates $\lambda, \tilde{\lambda}$), full (coordinates $x, \lambda, \tilde{\lambda}$) and dual (coordinates x, λ) spaces. We determine the action of dual conformal symmetry (inversion) in these spaces, in particular, we derive the transformations of the spinor variables λ and $\tilde{\lambda}$ from the known ones of x . We then show that all split-helicity tree-level gluon amplitudes (MHV as well as non-MHV) are manifestly dual conformal covariant. We conclude the section by recalling the anomalous dual conformal behavior of the loop corrections to the MHV gluon amplitudes.

In Sect. 3 we introduce on-shell superspace, parametrized by the spinor variables λ_α and $\tilde{\lambda}_{\dot{\alpha}}$, as well as by Grassmann variables η^A . We recall the content of the on-shell $\mathcal{N} = 4$ gluon supermultiplet and give its realization in this superspace. Implementing the conditions for on-shell supersymmetry, we rederive Nair's description of MHV tree-level superamplitudes. We then discuss the general structure of n -particle superamplitudes of the N^k MHV type (with $k = 0, 1, \dots, n - 4$) formulated in the on-shell superspace. We write them in a factorized form, with the MHV superamplitude as a prefactor, followed by a homogeneous polynomial of degree $4k$ in the η -variables.

In Sect. 4, by analogy with the bosonic case, we go from on-shell superspace to full superspace with coordinates $x_{\alpha\dot{\alpha}}, \lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \theta_\alpha^A, \eta^A$ and then to chiral dual superspace with coordinates $x_{\alpha\dot{\alpha}}, \lambda_\alpha, \theta_\alpha^A$. We describe the realization of the $\mathcal{N} = 4$ dual superconformal algebra $su(2, 2|4)$ in these spaces. This algebra has a central charge which is identified with helicity. We show that the MHV tree-level superamplitudes become manifestly dual superconformal covariant, if rewritten in the chiral dual superspace.

In Sect. 5 we apply this formalism to the simplest example of a NMHV superamplitude, the six-particle case. We propose a compact description of the $n = 6$ tree-level superamplitude as a product of the MHV superamplitude, followed by a set of three-point superconformal invariants. The generalization to one loop is achieved by turning the coefficients of the superinvariants into exactly dual conformal functions given in terms of one-loop box integrals. The dual conformal anomaly is then confined to the MHV prefactor. By expanding the superamplitude in the η -variables, we obtain explicit expressions for various gluon amplitudes, and show that they agree with the known results from the literature.

In Sect. 6 we generalize analysis to n -particle NMHV superamplitudes. We explain how to systematically construct the three-point superconformal invariants, which match the coefficients of the 3-mass-box integrals in the gluon amplitude. We show that their twistor coplanarity is an immediate corollary of dual supersymmetry, combined with the obvious property of 'super-coplanarity'. We then make a proposal for the complete one-loop NMHV superamplitude. As a byproduct, we obtain a new, very compact representation of the tree-level NMHV superamplitude, written down in a manifestly dual superconformal form.

²The results of this paper were reported at the workshops "Wonders of gauge theory", Paris/Saclay, June 2008 [32] and "Gauge theory and string theory", Zurich, July 2008.

Sect. 7 contains concluding remarks and outlook. Various technical details are collected in three appendices.

2 Dual conformal symmetry of gluon amplitudes

Since $\mathcal{N} = 4$ SYM is a (super)conformal theory, we can expect that the scattering amplitudes bear some traces of this symmetry. This is indeed true, as shown by Witten for tree-level MHV amplitudes in [1]. However, the action of the conformal group, being linear in configuration space, is rather complicated in momentum space. For example, the conformal boosts are generated by a second-order differential operator (see (2.10) below). After a twistor transform this action again becomes linear and the restrictions it imposes on the amplitude can be made manifest in the twistor representation. However, the inverse twistor transform is difficult to perform explicitly, therefore it is not easy to exploit this type of conformal symmetry.

The main subject of the present paper is a completely different kind of conformal symmetry of the scattering amplitudes, with linear action on the *momenta* of the particles. Its origin cannot be traced back to the Lagrangian of the theory. We may say that this is a *dynamical symmetry*. It becomes manifest after we introduce a *dual space* in Sect. 2.3. Its coordinates $x_{\alpha\dot{\alpha}}$ are expressed in terms of the particle momenta $p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}$, where λ_{α} and $\tilde{\lambda}_{\dot{\alpha}}$ are commuting spinor variables discussed in Sect. 2.1. In Sect. 2.4 we define a conformal group $SO(2,4)$ with the standard *linear action* on the dual space coordinates $x_{\alpha\dot{\alpha}}$ and derive the corresponding dual conformal transformations of the spinor variables λ_{α} and $\tilde{\lambda}_{\dot{\alpha}}$. In Sect. 2.5 we establish the transformation properties of the tree-level MHV gluon amplitudes under dual conformal transformations. In Sect. 2.6 we generalize the analysis to all split-helicity gluon amplitudes at tree level. In Sect. 2.7 we give the form of the dual conformal boost generator. Finally, in Sect. 2.8 we discuss the loop corrections to the MHV amplitude. They produce infrared divergences which break the dual conformal symmetry. This breakdown is controlled by an anomalous dual conformal Ward identity.

2.1 MHV gluon tree-level amplitudes

The main objects of study in this paper are scattering amplitudes of massless particles in gauge theories. Specifically, we are interested in $\mathcal{N} = 4$ SYM theory, which involves bosons (gluons and scalars), as well as fermions (gluinos). Each of these particles is characterized by its on-shell momentum p_i^{μ} ($i = 1 \dots n$) and helicity $h_i = \pm 1$ (gluons), $\pm 1/2$ (gluinos), 0 (scalars). We shall treat all particles as incoming, so that the total momentum conservation condition reads

$$\sum_{i=1}^n (p_i)^{\dot{\alpha}\alpha} = 0. \quad (2.1)$$

To solve the on-shell condition for p_i , it is convenient to introduce *commuting spinor* variables³

$$p_i^2 = \frac{1}{2}(p_i)^{\dot{\alpha}\alpha}(p_i)_{\alpha\dot{\alpha}} = 0 \quad \implies \quad (p_i)^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^{\alpha}, \quad (2.2)$$

³In this paper we use two-component Weyl spinor notation for Lorentz vectors and spinors, see Appendix A for details.

where λ_i^α ($\alpha = 1, 2$) and $\tilde{\lambda}_i^{\dot{\alpha}}$ ($\dot{\alpha} = \dot{1}, \dot{2}$) are two-component spinors. We will refer to the space with coordinates $(\lambda_i, \tilde{\lambda}_i)$ as the ‘on-shell’ space. If we wish to have real momenta p_i^μ in a space-time with Minkowskian signature, these spinors transform under the Lorentz group $SL(2, \mathbb{C})$, and $\tilde{\lambda} = \pm \lambda^*$. However, in various applications of the spinor formalism (for instance, in the generalized unitarity cuts approach of [22]), it is preferable to keep the momenta complex. In this case, $\tilde{\lambda}$ is not the complex conjugate of λ .

Equation (2.2) allows us to determine λ_i^α and $\tilde{\lambda}_i^{\dot{\alpha}}$ in terms of the (complex) momentum $(p_i)^{\dot{\alpha}\alpha}$ up to a *local* (i.e., depending on the point i) complex scale,

$$\lambda_i^\alpha \rightarrow c_i \lambda_i^\alpha, \quad \tilde{\lambda}_i^{\dot{\alpha}} \rightarrow c_i^{-1} \tilde{\lambda}_i^{\dot{\alpha}}. \quad (2.3)$$

For real momenta $(p_i)^{\dot{\alpha}\alpha}$ we have $|c_i| = 1$, and the resulting $U(1)$ phase can be identified with the particle helicity at point i . The standard convention is that the spinors λ and $\tilde{\lambda}$ carry helicities $-1/2$ and $+1/2$, respectively. Thus, the momentum $(p_i)^{\dot{\alpha}\alpha}$ has vanishing helicity. For complex momenta, we can still use the generalized notion of ‘helicity’, meaning the weight under the scale transformation (2.3).

Alternatively [1], we can say that the spinor variable $\lambda_{i\alpha}$ appears in the solution to the Weyl (or massless Dirac) equation for a chiral spin-1/2 particle:

$$(p_i)^{\dot{\alpha}\alpha} \psi_\alpha(p_i) = 0 \quad \implies \quad \psi_\alpha(p_i) = \lambda_{i\alpha} \Gamma(p_i), \quad (2.4)$$

provided the particle momentum satisfies the condition

$$(p_i)^{\dot{\alpha}\alpha} \lambda_{i\alpha} = 0. \quad (2.5)$$

The general solution to (2.5) for $(p_i)^{\dot{\alpha}\alpha}$ is of the form (2.2), thus introducing the antichiral spinor $\tilde{\lambda}_i^{\dot{\alpha}}$.

An important point is that the wave function $\psi_\alpha(p)$, belonging to a representation of the Lorentz group (a Weyl spinor), cannot have helicity. Indeed, helicity is a label for the *massless* representations of the Poincaré group, defined in a fixed Lorentz frame where $p^\mu = (p, 0, 0, p)$. The advantage of using the spinor variables λ and $\tilde{\lambda}$ is that we can make a bridge between Lorentz and massless Poincaré representations. Thus, the helicity $+1/2$ of the ‘particle’ (Poincaré state) $\Gamma(p)$ is compensated by the helicity $-1/2$ of λ_α , while its spinor index makes the wave function $\psi_\alpha(p)$ transform as a Lorentz representation. Similarly, we can relate an antichiral Weyl spinor field to a particle of helicity $-1/2$, $\bar{\psi}_{\dot{\alpha}}(p) = \tilde{\lambda}_{\dot{\alpha}} \bar{\Gamma}(p)$.

In the same way, we can introduce the spinor description of gluon states. On-shell gluons are massless Poincaré states of helicity ± 1 described, correspondingly, by $G^\pm(p)$. Their Lorentz covariant description makes use of the self-dual and anti-self-dual parts of the gluon field strength tensor, $G_{\alpha\beta}(p)$ and $\bar{G}_{\dot{\alpha}\dot{\beta}}(p)$, respectively, satisfying the equations of motion $p^{\dot{\alpha}\alpha} G_{\alpha\beta}(p) = \bar{G}_{\dot{\alpha}\dot{\beta}}(p) p^{\dot{\beta}\alpha} = 0$. Once again, the bridge between Poincaré and Lorentz representations is made with the help of the spinor variables, $G_{\alpha\beta}(p) = \lambda_\alpha \lambda_\beta G^+(p)$ and $\bar{G}_{\dot{\alpha}\dot{\beta}}(p) = \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\beta}} G^-(p)$.

Now, let us consider the simplest example of n -gluon MHV scattering amplitudes. By definition, they involve only two gluons of, say negative helicity, while the other $n - 2$ gluons have positive helicity.⁴ Different MHV gluon amplitudes are then defined by the positions of the

⁴For amplitudes with external particles different from gluons, their classification (MHV, NMHV, ...) is based on the total helicity weight (the sum of helicities of all particles). Namely, n -particle MHV amplitudes have the total helicity $n - 4$, next-to-MHV (NMHV) the helicity $n - 6$, etc.

two negative-helicity gluons, e.g., $(- - + \dots +)$, $(- + - + \dots +)$, etc. In the former case, the negative-helicity gluons appear contiguously and not separated by positive-helicity gluons as in the latter case. Such amplitudes are called ‘split-helicity amplitudes’ [28].

In the spinor formalism, the MHV tree-level amplitude with the two negative-helicity gluons occupying sites i and j is described by the following function of the spinor variables:

$$\mathcal{A}_{n;0}(1^+ \dots i^- \dots j^- \dots n^+) = i(2\pi)^4 \delta^{(4)}\left(\sum_{k=1}^n p_k\right) \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n 1 \rangle} \quad (2.6)$$

(here and in what follows $\mathcal{A}_{n;0}$ denotes a tree-level n -particle amplitude and i^\pm stands for the gluon state $G^\pm(p_i)$). The delta function in (2.6) is the solution of the momentum conservation condition (2.1), or equivalently of the condition for translation invariance of the amplitude,

$$p^{\dot{\alpha}\alpha} \mathcal{A}_n(p) = 0 \quad \implies \quad \mathcal{A}_n(p) \propto \delta^{(4)}\left(\sum_{i=1}^n p_i\right), \quad (2.7)$$

where the total momentum

$$p^{\dot{\alpha}\alpha} = \sum_{i=1}^n (p_i)^{\dot{\alpha}\alpha} \quad (2.8)$$

is the generator of translations in the momentum representation. The other factor in (2.6) is a rational and holomorphic function of the Lorentz invariant contraction of spinor variables (see Appendix A for our conventions for two-component spinors),

$$\langle i j \rangle = -\langle j i \rangle = \lambda_i^\alpha \epsilon_{\alpha\beta} \lambda_j^\beta = \lambda_i^\alpha \lambda_{j\alpha}. \quad (2.9)$$

Since each spinor λ carries helicity $(-1/2)$, this factor has helicity $(-1/2)$ at points i and j . By counting the degree of homogeneity in λ on the right-hand side of (2.6), we read off the helicities (-1) at points i and j , and $(+1)$ elsewhere.

We conclude this subsection by recalling that MHV tree-level amplitudes (2.6) have a conformal symmetry [1]. This is the ordinary conformal symmetry $SO(2,4)$ of the $\mathcal{N} = 4$ SYM Lagrangian, but realized on the particle momenta (or, equivalently, on the spinor variables λ and $\tilde{\lambda}$). In particular, the conformal boost generator takes the form of a second-order differential operator

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial^2}{\partial \lambda_i^\alpha \partial \tilde{\lambda}_i^{\dot{\alpha}}}, \quad (2.10)$$

leading to $k_{\alpha\dot{\alpha}} \mathcal{A}_{n;0} = 0$.

2.2 On-shell space and full space

The scattering amplitudes have the following general form

$$\mathcal{A}_n = i(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n p_i\right) A_n(p_1, \dots, p_n). \quad (2.11)$$

The function A_n depends on the momenta p_i of the incoming on-shell particles which are constrained in two ways. They are light-like vectors $p_i^2 = 0$ and satisfy the momentum conservation condition (2.1).

As we saw in the previous subsection, it is natural to solve the on-shell constraints by introducing spinor variables λ_i and $\tilde{\lambda}_i$, Eq. (2.2). For this reason we call them coordinates of the ‘on-shell space’. However, we should remember that the on-shell coordinates still satisfy a constraint following from momentum conservation (2.1):

$$\sum_{i=1}^n \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^{\alpha} = 0. \quad (2.12)$$

It means that not all of the variables $\lambda_i, \tilde{\lambda}_i$ are independent. For instance, we could eliminate any two spinors λ_k and λ_m (or two $\tilde{\lambda}$ ’s) in terms of the remaining $n - 2$.

Alternatively, we might wish to first solve the momentum conservation condition. To this end we introduce new ‘dual’ coordinates x_i (with $i = 1, \dots, n$)

$$\sum_{i=1}^n (p_i)^{\dot{\alpha}\alpha} = 0 \implies (p_i)^{\dot{\alpha}\alpha} = (x_i)^{\dot{\alpha}\alpha} - (x_{i+1})^{\dot{\alpha}\alpha}, \quad (2.13)$$

satisfying the cyclicity condition

$$x_{n+1} \equiv x_1. \quad (2.14)$$

Of course, we still have to impose the on-shell conditions $p_i^2 = 0$, which imply that the dual coordinates are constrained,

$$(x_i - x_{i+1})^2 = 0. \quad (2.15)$$

We must make it very clear that the new variables x_i have nothing to do with the coordinates in the original configuration space (which are the Fourier conjugates of the particle momenta p_i). Indeed, as can be seen from (2.13), the x -variables have the ‘wrong’ dimension of mass. As we will see shortly, these variables provide the natural framework for discussing the new ‘dual’ conformal symmetry of the amplitude.

At this stage we have proposed two sets of variables for describing the amplitude, the on-shell coordinates $\lambda_i, \tilde{\lambda}_i$ satisfying the constraint (2.12), and the dual coordinates x_i satisfying the constraint (2.15). We can now combine these two sets of variables into a single one, by defining the extended space with coordinates $(\lambda_i, \tilde{\lambda}_i, x_i)$, which we call the ‘full space’. From the compatibility of the solutions to (2.2) and (2.13) it follows that

$$(x_i - x_{i+1})^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^{\alpha}, \quad x_{n+1} \equiv x_1. \quad (2.16)$$

It is clear this identification yields both constraints (2.12) and (2.15).

We can think of the relation (2.16) as defining a surface in the full space. Then we can interpret the function $A_n(p_i) = A_n(\lambda_i, \tilde{\lambda}_i, x_i)$ appearing in the amplitude (2.11) as a function defined on this surface. Since A_n is a function of the particle momenta p_i , it can only depend on the dual coordinates through their differences $x_i - x_j = x_{ij}$, thus implying a dual translation invariance,

$$P_{\alpha\dot{\alpha}} A_n(\lambda_i, \tilde{\lambda}_i, x_i) = 0 \quad (2.17)$$

with $P_{\alpha\dot{\alpha}}$ being the generator of translations

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}. \quad (2.18)$$

We would like to stress the fact that the generator $P_{\alpha\dot{\alpha}}$ has nothing to do with the conventional translation generator $p_{\alpha\dot{\alpha}}$, Eq. (2.8). The latter acts as a shift of the coordinates in the original configuration space of the particles, while (2.18) acts as a shift of the *dual coordinates*.

To put it differently, we can say that the dual coordinates x_i can be solved for from (2.16), up to the freedom of choosing an arbitrary reference point, e.g., x_1 :

$$(x_i)^{\dot{\alpha}\alpha} = (x_1)^{\dot{\alpha}\alpha} - \sum_{k=1}^{i-1} \tilde{\lambda}_k^{\dot{\alpha}} \lambda_k^{\alpha}. \quad (2.19)$$

In other words, the definition of the dual coordinates (2.16) is invariant under shifts of the x_i 's by an arbitrary constant vector. Clearly, using the dual translation invariance (2.17) and the constraint (2.16), we can always eliminate x_i and obtain A_n as a function of λ_i and $\tilde{\lambda}_i$ on the on-shell space. The other possibility is to eliminate $\tilde{\lambda}_i$ with the help of (2.16) and to express A_n as a function of x_i and λ_i only. This leads to a holomorphic description of the amplitudes that we discuss in the following subsection.

2.3 Dual space

We can rewrite the constraint (2.16) without using the variables $\tilde{\lambda}_i$,

$$(x_{i \ i+1})^{\dot{\alpha}\alpha} \lambda_{i\alpha} = \lambda_i^{\alpha} (x_{i \ i+1})_{\alpha\dot{\alpha}} = 0, \quad (2.20)$$

where the shorthand notation $x_{i \ i+1} = x_i - x_{i+1}$ was used. This form of the constraints is reminiscent of, and partially inspired by the conditions on the spin-1/2 particle momentum (2.5). Additional motivation for introducing the constraints in the form of (2.20) will come from our discussion of dual supersymmetry in Sect. 4.2.

The relations (2.20) and (2.16) are in fact equivalent, since the general solution to (2.20) takes the form

$$(x_{i \ i+1})^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^{\alpha}, \quad (2.21)$$

thus introducing the *secondary* variables $\tilde{\lambda}_i$. In other words, the $\tilde{\lambda}$'s can be expressed in terms of x and λ by projecting (2.21) with, e.g., λ_{i+1}^{α} :

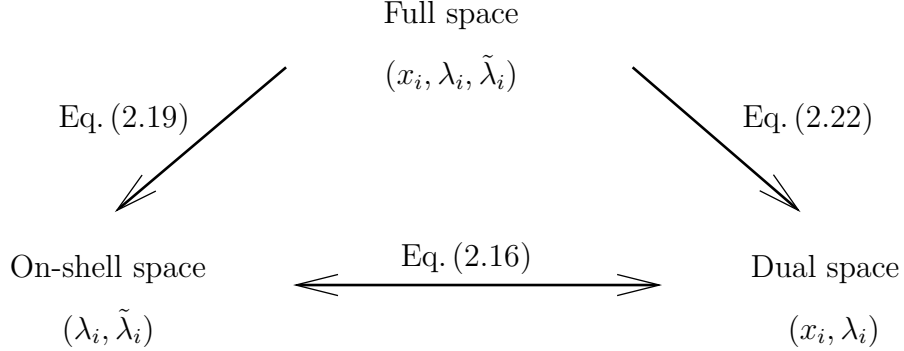
$$\tilde{\lambda}_i^{\dot{\alpha}} = \frac{(x_{i \ i+1})^{\dot{\alpha}\alpha} \lambda_{i+1\alpha}}{\langle i \ i+1 \rangle}. \quad (2.22)$$

Once we have deduced (2.21) from the defining constraint (2.20), we can make contact with the momenta of the particles through the identification $\tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^{\alpha} = (p_i)^{\dot{\alpha}\alpha}$. Applying the relation (2.22), the function $A_n(\lambda_i, \tilde{\lambda}_i, x_i)$ can now be regarded as a function of the variables x_i and λ_i .

We call the space with coordinates (x_i, λ_i) satisfying the defining constraint (2.20) the ‘dual space’. Note that this space is holomorphic – we only need the (complex) variables x_i, λ_i , but not their complex conjugates. Later on, in Sect. 4.2 we shall see that the holomorphic dual space has a natural extension to a chiral dual superspace.

In summary, we are proposing three different but equivalent descriptions of the scattering amplitudes in the on-shell space $(\lambda_i, \tilde{\lambda}_i)$, in the full space $(x_i, \lambda_i, \tilde{\lambda}_i)$ and in the dual space (x_i, λ_i) .

The relations between these equivalent descriptions are shown in the following diagram:



As we will see in a moment, the dual space offers the most convenient framework for understanding the new ‘dual conformal’ symmetry of the scattering amplitudes.

2.4 Dual conformal symmetry

As mentioned earlier, the main motivation for introducing the dual space was to exhibit a new, unexpected conformal symmetry of the MHV amplitude. This is the conformal group $SO(2, 4)$, which acts *linearly* on the dual space coordinates, i.e. on the particle momenta (and not, we stress again, on the coordinates of the particles in configuration space).

Our task in this section will be to learn how the dual conformal symmetry acts on the coordinates (x, λ) of the dual space. We shall assume that the dual coordinates $x^{\dot{\alpha}\alpha}$ transform in the standard way under the conformal group $SO(2, 4)$, and then deduce the transformation properties of λ^α by requiring that the defining relation (2.20) should remain covariant.

It is well known that the conformal group $SO(2, 4)$ can be obtained from the Poincaré group by adding the discrete operation of conformal inversion,

$$I[x_{\alpha\dot{\beta}}] = \frac{x_{\beta\dot{\alpha}}}{x^2} \equiv (x^{-1})_{\beta\dot{\alpha}}. \quad (2.23)$$

Notice that the inversion changes the chirality of two-component spinor indices. Performing an inversion, followed by an infinitesimal translation and then by another inversion, we obtain the generators of special conformal transformations (boosts), $K^\mu = IP^\mu I$. Then, commuting P^μ with K^ν we get the rest of the conformal algebra $o(2, 4)$, namely, Lorentz transformations and dilatations. The operation of inversion is an involution, $I^2 = \mathbb{I}$, which easily follows from the definition (2.23).

Let us now examine the effect of an inversion on the matrix $(x_i - x_{i+1})_{\alpha\dot{\beta}}$. Using (2.23) we obtain

$$I[(x_{ij})_{\alpha\dot{\beta}}] = (x_i^{-1} - x_j^{-1})_{\beta\dot{\alpha}} = -(x_j^{-1})_{\beta\dot{\beta}} (x_i - x_j)^{\dot{\beta}\gamma} (x_i^{-1})_{\gamma\dot{\alpha}} \equiv -(x_i^{-1} x_{ij} x_j^{-1})_{\beta\dot{\alpha}}. \quad (2.24)$$

Specifying to the case $j = i + 1$, this becomes

$$I[(x_{i\ i+1})_{\alpha\dot{\beta}}] = -(x_{i+1}^{-1})_{\beta\dot{\beta}} (x_{i\ i+1})^{\dot{\beta}\gamma} (x_i^{-1})_{\gamma\dot{\alpha}}. \quad (2.25)$$

Now, we can use (2.25) to deduce the inversion properties of the spinor variables λ , such that the constraint (2.20) remains covariant,

$$I[\lambda_i^\alpha (x_{i\ i+1})_{\alpha\dot{\beta}}] = -(x_{i+1}^{-1})_{\beta\dot{\beta}} (x_{i\ i+1})^{\dot{\beta}\gamma} (x_i^{-1})_{\gamma\dot{\alpha}} I[\lambda_i^\alpha] = 0. \quad (2.26)$$

Comparing this identity with the first relation in (2.20), we obtain that $(x_i^{-1})_{\gamma\dot{\alpha}} I[\lambda_i^\alpha] \sim \lambda_{i\gamma}$, or equivalently

$$I[\lambda_i^\alpha] = \kappa_i (x_i)^{\dot{\alpha}\beta} \lambda_{i\beta}, \quad I[\lambda_{i\alpha}] = \kappa_i \lambda_i^\beta (x_i)_{\beta\dot{\alpha}}, \quad (2.27)$$

where κ_i is an arbitrary (x -dependent) weight factor. Here the second relation follows from the first one after we raise/lower spinor indices and take into account the standard rules for inversion of the Levi-Civita tensors,⁵

$$I[\epsilon^{\alpha\beta}] = \epsilon^{\dot{\beta}\dot{\alpha}}, \quad I[\epsilon^{\dot{\alpha}\dot{\beta}}] = \epsilon^{\beta\alpha}. \quad (2.28)$$

It is straightforward to verify that the inversion defined in (2.27) is an involution, $I^2 = \mathbb{I}$, provided that $I[\kappa_i] = 1/\kappa_i$.

Using the transformation properties (2.27), we can easily show that the Lorentz invariant contraction of two adjacent spinors $\langle i \ i+1 \rangle$ is covariant under inversion:

$$I[\langle i \ i+1 \rangle] = I[\lambda_i^\alpha \lambda_{i+1\alpha}] = -\kappa_i \kappa_{i+1} \lambda_{i+1}^\beta (x_{i+1} x_i)_{\beta\alpha} \lambda_i^\alpha = \kappa_i \kappa_{i+1} \langle i \ i+1 \rangle_{x_{i+1}^2}, \quad (2.29)$$

where in the last relation we took into account the constraint (2.20) to replace $(x_{i+1} x_i)_{\beta\alpha} \lambda_i^\alpha = (x_{i+1} x_{i+1})_{\beta\alpha} \lambda_i^\alpha = x_{i+1}^2 \lambda_i^\beta$.

To fix the value of the weight factor κ_i , it is convenient to examine how the antichiral spinors $\tilde{\lambda}$ transform. We recall that they are not independent variables in the dual space, being related to x and λ through (2.22). Applying inversion to both sides of (2.22) and taking into account (2.25), (2.27) and (2.29), we find after some algebra that $\tilde{\lambda}$ also transform covariantly,

$$I[\tilde{\lambda}_i^{\dot{\alpha}}] = \tilde{\kappa}_i \tilde{\lambda}_{i\dot{\beta}} (x_i)^{\dot{\beta}\alpha}, \quad I[\tilde{\lambda}_{i\dot{\alpha}}] = \tilde{\kappa}_i (x_i)_{\alpha\dot{\beta}} \tilde{\lambda}_i^{\dot{\beta}}, \quad (2.30)$$

with $\tilde{\kappa}_i = (\kappa_i x_i^2 x_{i+1}^2)^{-1}$.

Let us now compare the relations (2.27) and (2.30). If we wish to treat $\tilde{\lambda}_i$ as the complex conjugate of λ_i , the choice of the weight factor κ_i is unambiguous (up to a phase), $\kappa_i = \tilde{\kappa}_i = (x_i^2 x_{i+1}^2)^{-1/2}$. We prefer instead to follow the holomorphic dual space description and treat $\tilde{\lambda}$ as secondary, dependent variables. In this case, we can take an arbitrary κ_i . A natural choice is $\kappa_i = 1/x_i^2$ and $\tilde{\kappa}_i = 1/x_{i+1}^2$. Substituting these expressions into (2.27) and (2.30), we obtain

$$I[\lambda_i^\alpha] = (x_i^{-1})^{\dot{\alpha}\beta} \lambda_{i\beta}, \quad I[\lambda_{i\alpha}] = \lambda_i^\beta (x_i^{-1})_{\beta\dot{\alpha}}, \quad (2.31)$$

$$I[\tilde{\lambda}_i^{\dot{\alpha}}] = \tilde{\lambda}_{i\dot{\beta}} (x_{i+1}^{-1})^{\dot{\beta}\alpha}, \quad I[\tilde{\lambda}_{i\dot{\alpha}}] = (x_{i+1}^{-1})_{\alpha\dot{\beta}} \tilde{\lambda}_{i+1}^{\dot{\beta}},$$

where $(x^{-1})_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}}/x^2$ and we applied the constraint (2.20) in the second line to replace $(x_i)_{\alpha\dot{\beta}} \tilde{\lambda}_i^{\dot{\beta}} = (x_{i+1})_{\alpha\dot{\beta}} \tilde{\lambda}_i^{\dot{\beta}}$.

We are now ready to discuss how to build covariants of the dual conformal transformations defined above. Specifically, for the purpose of constructing the scattering amplitudes, we are interested in Lorentz invariant functions of x_i , λ_i and $\tilde{\lambda}_i$. If we restrict ourselves to functions of x_i only, then the basic covariants are the ‘distances’ between two points x_{ij}^2 :⁶

$$I[x_{ij}^2] = \frac{x_{ij}^2}{x_i^2 x_j^2}. \quad (2.32)$$

⁵They can be obtained by considering, e.g., the Lorentz invariant contraction of two spinors, $\lambda_{1\alpha} \epsilon^{\alpha\beta} \lambda_{2\beta}$. Inversion changes the chirality of a spinor (see (2.27)), therefore $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\beta}\dot{\alpha}}$ have to be swapped.

⁶Recall that in the dual space for scattering amplitudes $x_{i+1}^2 = p_i^2 = 0$.

We have additional possibilities if we also include the spinor variables. Using the transformation properties (2.31), we can easily show that the Lorentz invariant contractions of two adjacent (anti)chiral spinors $\langle i \ i+1 \rangle \equiv \lambda_i^\alpha \lambda_{i+1\alpha}$ and $[i \ i+1] \equiv \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_{i+1}^{\dot{\alpha}}$ are covariant under inversion (see also (2.29)):

$$\begin{aligned} I \left[\langle i \ i+1 \rangle \right] &= (x_i^2)^{-1} \langle i \ i+1 \rangle, \\ I \left[[i \ i+1] \right] &= (x_{i+2}^2)^{-1} [i \ i+1]. \end{aligned} \quad (2.33)$$

Note, however, that all other Lorentz invariant contractions $\langle ij \rangle$ (or $[ij]$) with $j \neq i+1$ are *not covariant* under the dual conformal symmetry. Besides these simplest examples, there exists a large variety of dual conformally covariant and Lorentz invariant ‘strings’ made of x_i , λ_i and $\tilde{\lambda}_i$ that will be presented in Sect. 2.6.

2.5 Dual conformal transformation properties of the MHV tree-level gluon amplitudes

In this section we examine the transformation properties of the MHV tree-level gluon amplitudes (2.6) under the dual conformal transformations (2.23) and (2.31).

Let us first apply (2.23) and (2.31) to the MHV amplitude $\mathcal{A}_{n;0}^{\text{MHV}}(1^- 2^- 3^+ \dots n^+)$, which is an example of a split-helicity amplitude. According to (2.6), this amplitude is given by a product of a momentum delta function and a rational holomorphic function of the spinor variables with $i=1$ and $j=2$. We start with the latter and use (2.33) to obtain its conformal inversion,

$$I \left[\frac{\langle 1 \ 2 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \dots \langle n \ 1 \rangle} \right] = \frac{x_2^2 x_3^2 \dots x_n^2}{(x_1^2)^3} \frac{\langle 1 \ 2 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \dots \langle n \ 1 \rangle}. \quad (2.34)$$

Note the appearance of conformal weights (-3) at point x_1 and $(+1)$ at all remaining points.⁷

Let us now examine the conformal properties of the delta function in (2.6). Its role is to impose the momentum conservation condition (2.1), or equivalently (2.13). Any function, multiplied by $\delta^{(4)}(\sum_{i=1}^n p_i)$ (for example, the factor discussed in (2.34)) is thus defined on the constraint surface (2.16) in the full space with the coordinates $(x_i, \lambda_i, \tilde{\lambda}_i)$. By construction, the transformation properties (2.23) and (2.31) are consistent with the constraints (2.13) and (2.16), therefore we may say that dual conformal transformations do not take us out of the constraint surface. But we still have to answer the question how the delta function itself transforms. Obviously, we cannot use the constraints (2.13), together with the cyclicity condition $x_{n+1} = x_1$, because this will lead to the vanishing of the argument of the delta function.

The answer to this question is found by realizing that the momentum conservation constraint is solved by the substitution $p_i = x_i - x_{i+1}$ only if the cyclicity condition (2.14) is imposed. Let us relax it for a moment, $x_{n+1} \neq x_1$, while still keeping the relations $(x_i - x_{i+1})^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha$. In particular, we assume that $(x_n - x_{n+1})^{\dot{\alpha}\alpha} = \tilde{\lambda}_n^{\dot{\alpha}} \lambda_n^\alpha$ instead of the cyclic $(x_n - x_1)^{\dot{\alpha}\alpha} = \tilde{\lambda}_n^{\dot{\alpha}} \lambda_n^\alpha$.

⁷Here we use the same conventions for assigning conformal weights as in conformal field theory. Namely, the two-point function of a primary scalar field with conformal weight j has the form $\langle \phi(x_1) \phi(x_2) \rangle = 1/x_{12}^{2j}$. Under inversions it transforms as $I[\langle \phi(x_1) \phi(x_2) \rangle] = (x_1^2 x_2^2)^j \langle \phi(x_1) \phi(x_2) \rangle$, which explains the weight assignments in (2.34).

Then $\sum_{i=1}^n p_i = x_1 - x_{n+1} \neq 0$, the delta function in (2.6) is replaced by $\delta^{(4)}(x_1 - x_{n+1})$ and the amplitude takes the following form:

$$\mathcal{A}_{n;0}^{\text{MHV}}(1^- 2^- 3^+ \dots n^+) = i(2\pi)^4 \delta^{(4)}(x_1 - x_{n+1}) \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n 1 \rangle}. \quad (2.35)$$

The role of the delta function now is to impose the identification $x_{n+1} = x_1$ in the rational factor in (2.6), instead of the momentum conservation (2.1). Such a delta function, defined in the dual space, is manifestly conformally covariant. It has conformal weight four at point 1 (needed to compensate that of the integration measure, $\int d^4 x_1 \delta^{(4)}(x_1 - x_{n+1}) = 1$). Thus, the tree-level MHV split-helicity amplitude transforms under conformal inversions as follows,

$$I[\mathcal{A}_{n;0}^{\text{MHV}}(1^- 2^- 3^+ \dots n^+)] = (x_1^2 x_2^2 \dots x_n^2) \mathcal{A}_{n;0}^{\text{MHV}}(1^- 2^- 3^+ \dots n^+), \quad (2.36)$$

so we conclude that it has conformal weight (+1) at all points. In a similar manner, it is straightforward to verify that all tree-level MHV split-helicity amplitudes $\mathcal{A}_{n;0}^{\text{MHV}}(\dots G_i^- G_{i+1}^- \dots)$ are dual conformal covariant. Moreover, as we will show in the next subsection, the same property holds for *all split-helicity tree-level non-MHV amplitudes* $\mathcal{A}_{n;0}(1^- \dots q^-(q+1)^+ \dots n^+)$, i.e. those amplitudes in which the negative-helicity gluons appear contiguously.

However, the tree-level non-split-helicity MHV amplitudes $\mathcal{A}_{n;0}^{\text{MHV}}(\dots i^- \dots j^- \dots)$, Eq. (2.6), involve the spinor contractions $\langle i j \rangle$ which, as noted above, are not covariant for $|i - j| > 1$. Therefore the MHV gluon amplitudes where the negative-helicity gluons are not at adjacent points, are not dual conformal, at least not on their own. At first sight, it might seem that dual conformal symmetry is an isolated property of a very special class of gluon amplitudes, the split-helicity amplitudes. In fact, the full understanding of the role of dual conformal symmetry is achieved when the gluon amplitudes are combined together with the amplitudes involving other particles (gluinos, scalars) into a bigger and unifying object, the $\mathcal{N} = 4$ superamplitude. We do this in Sect. 4.4 where we show that only the complete MHV superamplitude (see (3.22) below) has well defined conformal properties. There we also explain why its various components, i.e. the split-helicity and other gluon amplitudes behave differently under dual conformal transformations.

2.6 Split-helicity gluon amplitudes at tree level

Let us now show that dual conformality is a property of a much wider class of non-MHV amplitudes, the split-helicity amplitudes. These are color-ordered n -gluon amplitudes with the helicities distributed as $\mathcal{A}_{n;0}(1^- \dots q^-, (q+1)^+ \dots n^+)$.

The split-helicity amplitudes are special because they form a closed set under the BCF/BCFW tree-level recursion relations [33, 34]. These relations have been solved in [28] and the explicit expression for the tree level split-helicity amplitudes reads

$$\mathcal{A}_{n;0}(1^- \dots q^-(q+1)^+ \dots n^+) = i(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n p_i\right) \sum_{k=0}^{\min(q-3, n-q-2)} \sum_{A_k, B_{k+1}} \frac{N_1^3 N_2 N_3}{D_1 D_2 D_3}. \quad (2.37)$$

Here $A_k = (a_1, a_2, \dots, a_k)$ and $B_{k+1} = (b_1, b_2, \dots, b_{k+1})$ range over all subsets of the indices $\{2, \dots, q-2\}$ and $\{q+1, \dots, n-1\}$ of size k and $k+1$ respectively. In terms of the dual variables, the quantities

N_i are given by,

$$\begin{aligned} N_1 &= \langle 1 | x_{1, b_1+1} x_{b_1+1, a_1+1} x_{a_1+1, b_2+1} \cdots x_{b_{k+1}+1, q} | q \rangle, \\ N_2 &= \langle b_1 + 1, b_1 \rangle \cdots \langle b_{k+1} + 1, b_{k+1} \rangle, \\ N_3 &= [a_1, a_1 + 1] \cdots [a_k, a_k + 1], \end{aligned} \quad (2.38)$$

and similarly for D_i ,

$$\begin{aligned} D_1 &= x_{2, b_1+1}^2 x_{b_1+1, a_1+1}^2 x_{a_1+1, b_2+1}^2 \cdots x_{b_{k+1}+1, q}^2, \\ D_2 &= [2, 3][4, 5] \cdots [q-2, q-1] \langle q, q+1 \rangle \langle q+1, q+2 \rangle \cdots \langle n, 1 \rangle, \\ D_3 &= [2 | x_{2, b_1+1} | b_1 + 1 \rangle \langle b_1 | x_{b_1, a_1} | a_1 \rangle [a_1 + 1 | x_{a_1+1, b_2+1} | b_2 + 1 \rangle \cdots \langle b_{k+1} | x_{b_{k+1}, q-1} | q-1 \rangle. \end{aligned} \quad (2.39)$$

The important property of the relations (2.38) and (2.39) is that the quantities N_i and D_i are built from manifestly dual conformal covariant objects like x_{ij}^2 , $\langle i | i+1 \rangle$, $[i, i+1]$ as well as strings of x 's 'sandwiched' between λ 's and $\tilde{\lambda}$'s. The shortest string of the latter type is⁸

$$\lambda_i^\alpha (x_{ij})_{\alpha\dot{\alpha}} \tilde{\lambda}_i^{\dot{\alpha}} \equiv \langle i | x_{ij} | j \rangle = \langle i | x_{i+1, j} | j \rangle = \langle i | x_{i+1, j-1} | j \rangle, \quad (2.40)$$

where we have used the identities, e.g., $x_{ij} = x_{i, i+1} + x_{i+1, j}$ and $\langle i | x_{i, i+1} = \langle ii \rangle [i] \equiv 0$. The first way of writing this string, $\langle i | x_{ij} | j \rangle$, clearly shows that it is dual conformally covariant. Indeed, from (2.24) and (2.31) we obtain

$$I \left[\langle i | x_{ij} | j \rangle \right] = \langle i | x_i^{-1} \cdot (x_i^{-1} x_{ij} x_j^{-1}) \cdot \frac{x_j}{x_{j+1}^2} | j \rangle = \frac{\langle i | x_{ij} | j \rangle}{x_i^2 x_{j+1}^2}. \quad (2.41)$$

Further, longer strings can be formed by multiplying together several x -matrices. For example,

$$I \left[\langle i | x_{ij} x_{jk} | k \rangle \right] = \frac{\langle i | x_{ij} x_{jk} | k \rangle}{x_i^2 x_j^2 x_k^2}. \quad (2.42)$$

It is straightforward to generalize this to strings built from an arbitrary number of x insertions. It is sufficient that the two neighboring x 's have a common subscript to ensure that the entire string transforms covariantly. This is exactly what we see in (2.38) and (2.39).

Performing conformal inversion in (2.38) and (2.39) and combining the various conformal weight factors, we find that the ratio of N - and D -functions entering (2.37) has conformal weight (+1) at all points except for point x_1 which has weight (-3). Similarly to the MHV split-helicity amplitudes, the delta function $\delta^{(4)}(\sum_{i=1}^n p_i) = \delta^{(4)}(x_1 - x_{n+1})$ is conformally covariant, bringing in an additional conformal weight (+4) at point x_1 , thus making the total weight equal to (+1). Since the assignment of dual conformal weights is independent of A_k and B_{k+1} , every term in the sum (2.37) has the same weight (+1) at all points and, therefore, the whole expression for the split-helicity amplitude (2.37) is manifestly dual conformal covariant,

$$I \left[\mathcal{A}_{n;0} (1^- \cdots q^-(q+1)^+ \cdots n^+) \right] = (x_1^2 x_2^2 \cdots x_n^2) \mathcal{A}_{n;0} (1^- \cdots q^-(q+1)^+ \cdots n^+) . \quad (2.43)$$

⁸Clearly, this string vanishes for $j = i + 1$.

2.7 Dual conformal boost generators in the full space

As discussed above, the form of the generators of infinitesimal dual conformal transformations can be obtained through the relation $K = IPI$. An alternative, more geometrical approach is to consider the full space with all the coordinates $x, \lambda, \tilde{\lambda}$. In this space the amplitude has support only on a surface defined by the constraints (2.16).

To discuss the infinitesimal transformation properties of the amplitude under dual conformal symmetry we need to construct generators which preserve the surface. This is achieved by complementing the conformal generators acting on the x coordinates,

$$\sum_{i=1}^n x_i^{\dot{\alpha}\beta} x_i^{\dot{\beta}\alpha} \frac{\partial}{\partial x_i^{\dot{\beta}\beta}}, \quad (2.44)$$

by terms which act on λ and $\tilde{\lambda}$ in such a way that they commute with the constraints modulo constraints. There is some ambiguity in this procedure, just as we saw that there is an ambiguity in defining the inversions of λ and $\tilde{\lambda}$ in Sect. 2.4. The choice which corresponds to the inversion defined there is

$$K^{\dot{\alpha}\alpha} = \sum_{i=1}^n \left[x_i^{\dot{\alpha}\beta} x_i^{\dot{\beta}\alpha} \frac{\partial}{\partial x_i^{\dot{\beta}\beta}} + x_i^{\dot{\alpha}\beta} \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\beta} + x_{i+1}^{\dot{\beta}\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} \right]. \quad (2.45)$$

This formula summarizes the infinitesimal dual conformal transformation of all variables in the full space with coordinates $x_i, \lambda_i, \tilde{\lambda}_i$. To derive the action of the dual conformal generator in the on-shell space with coordinates $\lambda_i, \tilde{\lambda}_i$ we can simply ignore the first term in (2.45). Note however that the action of $K^{\dot{\alpha}\alpha}$ on a function of $\lambda_i, \tilde{\lambda}_i$ will necessarily introduce the dual coordinates x_i , so the on-shell space is not best suited for investigating the dual conformal properties of amplitudes. To derive the action of the conformal generator in the dual space with coordinates x_i, λ_i we can ignore the third term in (2.45).

2.8 Dual conformal properties of the complete MHV amplitude

The perturbative (loop) corrections to the tree-level amplitude (2.6) take the form

$$\mathcal{A}^{\text{MHV}} = \mathcal{A}_{n;0}^{\text{MHV}} M_n(x_i) \quad (2.46)$$

where

$$M_n(x_i) = 1 + a M_n^{(1)}(x_i) + a^2 M_n^{(2)}(x_i) + \dots \quad (2.47)$$

is a function of the dual coordinates x_i (or, equivalently, of the momenta p_i)⁹, given by its perturbative expansion in terms of the coupling $a = g^2 N / 8\pi^2$. Each term in this expansion is a combination of divergent loop momentum integrals. For example, in the dimensional regularization scheme ($D = 4 - 2\epsilon$ with $\epsilon > 0$, regularization scale μ) the one-loop correction is given by the function [31]

$$M_n^{(1)}(x_i) = -\frac{1}{\epsilon^2} \sum_{i=1}^n (-x_{i,i+2}^2 \mu^2)^\epsilon + F_n^{(1)} \quad (2.48)$$

⁹The reason why there is no explicit dependence on the spinor variables $\lambda, \tilde{\lambda}$ in the function M is that the helicity weights of the amplitude are carried by the tree-level prefactor in (2.46). We know that $\lambda_i, \tilde{\lambda}_i$ can be expressed in terms of $x_{i,i+1}$ from (2.21), up to a helicity scale. Then any helicity neutral function $f(\lambda, \tilde{\lambda})$ can be rewritten as a function of x .

where

$$F_n^{(1)} = \frac{1}{2} \sum_{i=1}^n g_{n,i}, \quad g_{n,i} = - \sum_{r=2}^{\lfloor \frac{n}{2} \rfloor - 1} \ln \left(\frac{x_{i,i+r}^2}{x_{i,i+r+1}^2} \right) \ln \left(\frac{x_{i+1,i+r+1}^2}{x_{i,i+r+1}^2} \right) + D_{n,i} + L_{n,i} + \frac{3}{2} \zeta_2. \quad (2.49)$$

For n even, $n = 2m$, the functions $D_{n,i}$ and $L_{n,i}$ are

$$D_{n,i} = - \sum_{r=2}^{m-2} \text{Li}_2 \left(1 - \frac{x_{i,i+r}^2 x_{i-1,i+r+1}^2}{x_{i,i+r+1}^2 x_{i-1,i+r}^2} \right) - \frac{1}{2} \text{Li}_2 \left(1 - \frac{x_{i,i+m-1}^2 x_{i-1,i+m}^2}{x_{i,i+m}^2 x_{i-1,i+m-1}^2} \right), \quad (2.50)$$

$$L_{n,i} = \frac{1}{4} \ln^2 \left(\frac{x_{i,i+m}^2}{x_{i+1,i+m+1}^2} \right)$$

and for n odd there are similar expressions.

Clearly, the presence of divergences and consequently the need to use dimensional regularization breaks dual conformal invariance. Remarkably, however, this breakdown occurs in a controlled way to all orders in the coupling, as we have shown in [11, 12]. To see this one splits $\ln M_n = \ln Z_n + \ln F_n$, where $\ln Z_n$ contains the infrared divergences (double and simple poles), while the finite part $\ln F_n$ is subject to the anomalous conformal Ward identity

$$K^\mu \ln F_n = \sum_{i=1}^n (2x_i^\nu x_i \cdot \partial_i - x_i^2 \partial_i^\nu) \ln F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^n \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\nu. \quad (2.51)$$

The operator on the left-hand side is the generator of dual conformal boosts K^μ , obtained by applying (2.45) to a function of the dual coordinates x_i only. The anomaly term on the right-hand side is determined by the cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$.

The general solution of (2.51) allows some freedom in the form of an arbitrary function of the conformally invariant cross-ratios

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{jk}^2}, \quad (2.52)$$

if $n \geq 6$ (for $n = 4, 5$ there exist no cross-ratios, due to the light-like separation of adjacent points).

In summary, the MHV superamplitude consists of two factors, the tree-level prefactor $\mathcal{A}_{n;0}^{\text{MHV}}$ and the perturbative corrections factor $M_n(x_i)$. The former is an exact dual conformal covariant, while the latter has an anomalous dual conformal behavior controlled by an all-order Ward identity. We can conclude that the MHV superamplitude is compatible with dual conformal symmetry, after taking the anomaly into account.

3 $\mathcal{N} = 4$ supersymmetry and scattering amplitudes

Apart from the two gluon states G^\pm with helicities ± 1 , the $\mathcal{N} = 4$ SYM theory also describes eight fermion states (gluinos) Γ_A and $\bar{\Gamma}^A$ with helicities $1/2$ and $-1/2$, respectively, and six scalars (helicity zero states) $S_{AB} = -S_{BA}$. Here $A, B, C, D = 1, \dots, 4$ are indices of the (anti)fundamental representation of the R symmetry group $SU(4)$. These particles can scatter into each other in

many different combinations, which results in a large variety of amplitudes. For instance, the MHV tree amplitude involving one negative-helicity gluon and two gluinos reads:¹⁰

$$\mathcal{A}_{n;0}(G^-(1)\Gamma_A(2)\bar{\Gamma}^B(3)G^+(4)\dots G^+(n)) = i(2\pi)^4\delta_A^B \delta^{(4)}\left(\sum_{i=1}^n p_i\right) \frac{\langle 12\rangle\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}. \quad (3.1)$$

These various scattering amplitudes are related to each other through supersymmetric Ward identities [35, 36].

To discuss the symmetry properties of the scattering amplitudes, it would be desirable to find a way to present all scattering amplitudes in the $\mathcal{N} = 4$ theory as one simple and compact object with manifest supersymmetry. In the simplest case of MHV tree-level amplitudes this has been achieved some time ago by Nair [29]¹¹ who proposed to use a particular type of $\mathcal{N} = 4$ on-shell superspace.

In this section we rederive the well-known expression for the MHV superamplitude by exploiting the on-shell supersymmetry. Then we generalize the construction to an arbitrary n -particle superamplitude. We start with a brief reminder of the structure of the $\mathcal{N} = 4$ supermultiplets of massless states in Sect. 3.1. Then we reformulate these multiplets in on-shell (or light-cone) superspace in Sect. 3.2. This superspace is used in Sect. 3.3 to give a general description of all n -particle superamplitudes, including Nair's MHV amplitude, but also all non-MHV amplitudes. These superamplitudes are expressed in terms of invariants of the on-shell supersymmetry.

3.1 $\mathcal{N} = 4$ gluon supermultiplet

Here we recall (see, e.g., the textbook [37]) how one builds the massless representations (or supermultiplets) of the $\mathcal{N} = 4$ supersymmetry algebra

$$\{q_\alpha^A, \bar{q}_{B\dot{\alpha}}\} = \delta_B^A p_{\alpha\dot{\alpha}} \equiv \delta_B^A p^\mu (\sigma_\mu)_{\alpha\dot{\alpha}}, \quad (3.2)$$

where $\sigma_\mu = (\mathbb{I}, \vec{\sigma})$ and $\vec{\sigma}$ are the Pauli matrices. In the massless case, $p_\mu^2 = 0$, we can choose the Lorentz frame in which $p^\mu = (p, 0, 0, p)$ and the relation (3.2) becomes

$$\{q_\alpha^A, \bar{q}_{B\dot{\alpha}}\} = \delta_B^A (1 + \sigma_3)_{\alpha\dot{\alpha}} p, \quad (3.3)$$

so the algebra (3.3) is reduced to the Clifford algebra

$$\{q_1^A, \bar{q}_{B\dot{1}}\} = 2\delta_B^A p \quad (3.4)$$

with all the other anticommutators vanishing. In this frame the states (massless Poincaré representations) are labeled by their helicity, the eigenvalue of the Lorentz generator J_{12} . For chiral spinors it is $\frac{1}{2}(\sigma_{12})_\alpha^\beta$, and the helicity of, e.g., q_1^A is $1/2$. For antichiral spinors $J_{12} = \frac{1}{2}(\tilde{\sigma}_{12})_{\dot{\alpha}}^{\dot{\beta}}$, so that the helicity of $\bar{q}_{A\dot{1}}$ is $-1/2$.

Next, we define a vacuum state of helicity h by the condition that it be annihilated by all those supersymmetry generators which anticommute among themselves (annihilation operators):

$$q_1^A|h\rangle = q_2^A|h\rangle = \bar{q}_{A\dot{2}}|h\rangle = (J_{12} - h)|h\rangle = 0. \quad (3.5)$$

¹⁰It has total helicity $n - 4$, therefore it is an MHV amplitude.

¹¹Here we use a modified version of Nair's description proposed by Witten [1].

Then the massless supermultiplet of states is obtained by applying the four creation operators $\bar{q}_{A\dot{1}}$ to the vacuum:

State	Helicity	Multiplicity	
$ h\rangle$	h	1	
$\bar{q}_{A\dot{1}} h\rangle$	$h - 1/2$	4	
$\bar{q}_{A\dot{1}}\bar{q}_{B\dot{1}} h\rangle$	$h - 1$	6	
$\epsilon^{ABCD}\bar{q}_{A\dot{1}}\bar{q}_{B\dot{1}}\bar{q}_{C\dot{1}} h\rangle$	$h - 3/2$	4	
$\epsilon^{ABCD}\bar{q}_{A\dot{1}}\bar{q}_{B\dot{1}}\bar{q}_{C\dot{1}}\bar{q}_{D\dot{1}} h\rangle$	$h - 2$	1	(3.6)

In a physical theory the helicity should be $|h| \leq 2$, so in the case $\mathcal{N} = 4$ the allowed values are $h = 1, 3/2, 2$. We see that the multiplet obtained by choosing $h = 1$ is self-conjugate under PCT, since it contains all the helicities ranging from $+1$ to -1 . This is the so-called $\mathcal{N} = 4$ gluon supermultiplet, describing massless particles of helicities ± 1 (gluons), $\pm 1/2$ (gluinos) and 0 (scalars).

3.2 Covariant on-shell $\mathcal{N} = 4$ superspace

The construction of the preceding section has the drawback that it requires the choice of a special frame, thus manifestly breaking Lorentz invariance. Having the spinor variables λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ at our disposal, we can do better. We can reproduce the supermultiplet (3.6) in a *manifestly Lorentz covariant way*.¹²

Let us rewrite the supersymmetry algebra (3.2) using the representation (2.2) of the on-shell momentum:

$$\{q_\alpha^A, \bar{q}_{B\dot{\alpha}}\} = \delta_B^A \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \quad (3.7)$$

The two-component spinor q_α^A has two Lorentz covariant projections, one ‘parallel’ to λ_α , $q_{\parallel\alpha}^A = \lambda_\alpha q^A$ (with $\lambda^\alpha q_{\parallel\alpha}^A = 0$), the other ‘orthogonal’, $q_\perp^A = \lambda^\alpha q_\alpha^A$. The same applies to $\bar{q}_{A\dot{\alpha}}$. Multiplying (3.7) by λ^α or by $\tilde{\lambda}^{\dot{\alpha}}$, we see that the projections q_\perp^A and $\bar{q}_{\perp A}$ anticommute with each other and with the rest of the generators. These are the covariant analogs of the explicit light-cone projections q_2^A and $\bar{q}_{A\dot{2}}$ from (3.5). Then we substitute the projections $q_{\parallel\alpha}^A$ and $\bar{q}_{\parallel A\dot{\alpha}}$ in (3.7) and obtain

$$\{q^A, \bar{q}_B\} = \delta_B^A. \quad (3.8)$$

Clearly, this is the covariant analog of the Clifford algebra (3.4), with q^A and \bar{q}_A being the equivalents of the annihilation operator q_1^A and the creation operator $\bar{q}_{A\dot{1}}$, respectively.

It is well known that such algebras are most naturally realized in terms of anticommuting (Grassmann) variables η^A :

$$q^A = \eta^A, \quad \bar{q}_A = \frac{\partial}{\partial \eta^A}, \quad \{\eta^A, \eta^B\} = 0. \quad (3.9)$$

Since the creation operator $\bar{q}_{A\dot{1}}$ has helicity $-1/2$, the variables η^A should have helicity $1/2$.

¹²The idea to use auxiliary commuting spinor variables for a covariant description of light-cone supersymmetry has been introduced a long time ago under the name of ‘light-cone harmonic superspace’ [38].

We can now use the generators (3.9) to reproduce the content of the multiplet (3.6) in the convenient and compact form of a super-wave function

$$\begin{aligned}\Phi(p, \eta) = & G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2} \eta^A \eta^B S_{AB}(p) + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p) \\ & + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p).\end{aligned}\quad (3.10)$$

The analog of the vacuum with helicity $h = 1$ is the first term in (3.10), which can be identified as $G^+(p) = \Phi(p, 0)$. The next state in the multiplet is obtained by applying the creation operator \bar{q}_A , i.e. $\Gamma_A(p) = \bar{q}_A \Phi(p, \eta)|_{\eta=0}$, etc. Notice that the helicity of each component wave function in (3.10) is balanced by that of the Grassmann variables, so that the super-wave function $\Phi(p, \eta)$ carries overall helicity (+1).

Let us recall the discussion of particle states and wave functions from Sect. 2.1. There we used the spinor variables to relate the Lorentz covariant wave function of, e.g., a gluon $G_{\alpha\beta}(p)$ to the corresponding state of helicity +1, $G_{\alpha\beta}(p) = \lambda_\alpha \lambda_\beta G^+(p)$. Now we see all these states, i.e. wave functions ‘stripped’ of their Lorentz structures, gathered together in the super-wave function (3.10).

To summarize, with the help of the spinor variables we have been able to covariantly split the supersymmetry generators into two halves (the covariant analogs of the light-cone projections). The projections q_\perp^A and $\bar{q}_{\perp A}$ play no role in the construction of the massless supermultiplet, therefore we can set them to zero. Then, the on-shell supersymmetry generators are realized on the light-cone super-wave functions (3.10) as follows:

$$q_\alpha^A = \lambda_\alpha \eta^A, \quad \bar{q}_{A\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta^A}, \quad (3.11)$$

so that under infinitesimal supersymmetry transformations we find

$$\delta\Phi(p, \eta^A) = (\epsilon_A^\alpha q_\alpha^A + \bar{\epsilon}^{A\dot{\alpha}} \bar{q}_{A\dot{\alpha}}) \Phi(p, \eta^A) = \left(\epsilon_A \eta^A + \bar{\epsilon}^A \frac{\partial}{\partial \eta^A} \right) \Phi(p, \eta^A). \quad (3.12)$$

Note that the super-wave function undergoes transformations with *covariantly projected* parameters $\epsilon_A \equiv \epsilon_A^\alpha \lambda_\alpha$ and $\bar{\epsilon}^A \equiv \bar{\epsilon}^{A\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}}$. In this way we obtain the supersymmetry transformations of the component wave functions

$$\delta G^+ = \bar{\epsilon}^A \Gamma_A, \quad \delta \Gamma_A = \epsilon_A G^+ - \bar{\epsilon}^B S_{BA}, \quad \text{etc.} \quad (3.13)$$

Finally, we point out that our approach to the on-shell superspace is holomorphic. We made our choice in favor of the Grassmann variables η^A , and did not use their conjugates $\bar{\eta}_A$. Equivalently, we can say that here we favor a chiral description, since in (3.11) we chose to represent the chiral generator q_α^A as a multiplication operator, and not the antichiral $\bar{q}_{A\dot{\alpha}}$. This choice will subsequently determine our preference for a chiral dual superspace in Sect. 4.2. Of course, we could have equally well described the $\mathcal{N} = 4$ gluon supermultiplet by an anti-holomorphic super-wave function of overall helicity (−1):

$$\begin{aligned}\bar{\Phi}(\bar{\eta}, p) = & G^-(p) + \bar{\eta}_A \bar{\Gamma}^A(p) + \frac{1}{2} \bar{\eta}_A \bar{\eta}_B \bar{S}^{AB}(p) + \frac{1}{3!} \bar{\eta}_A \bar{\eta}_B \bar{\eta}_C \epsilon^{ABCD} \Gamma_D(p) \\ & + \frac{1}{4!} \bar{\eta}_A \bar{\eta}_B \bar{\eta}_C \bar{\eta}_D \epsilon^{ABCD} G^+(p).\end{aligned}\quad (3.14)$$

We see once again the special property of this multiplet of being PCT self-conjugate. In fact, this is the reason why we can choose a purely holomorphic description of the multiplet and, subsequently, a chiral dual superspace for the superamplitude. In a theory with $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry the gluon multiplet is not self-conjugate, therefore we would need both a holomorphic *and* an anti-holomorphic super-wave functions for the full theory.

The equivalence of the two descriptions of the $\mathcal{N} = 4$ gluon multiplet can also be shown by establishing an explicit relation between (3.10) and (3.14). It takes the form of a Grassmann Fourier transform:

$$\bar{\Phi}(p, \bar{\eta}) = \int d^4\eta \, e^{\bar{\eta}^A \eta^A} \, \Phi(p, \eta). \quad (3.15)$$

3.3 Superamplitudes in $\mathcal{N} = 4$ SYM

In this section we construct a superamplitude which gives a compact description of the scattering amplitudes of all the particles in the $\mathcal{N} = 4$ theory,

$$\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \mathcal{A}(\Phi(1)\Phi(2)\dots\Phi(n)), \quad (3.16)$$

where $\Phi(i) = \Phi(p_i, \eta_i)$ (with $i = 1, \dots, n$) stands for the $\mathcal{N} = 4$ supermultiplet (3.10) and $(p_i)^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha$ is the on-shell momentum of the particles in the supermultiplet. In general, $\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i)$ is an inhomogeneous polynomial of degree $4n$ in the odd variables η_i^A . However, as we will see shortly, invariance under on-shell supersymmetry restricts the minimal degree to be 8 and the maximal to be $4n - 8$. How can we construct such invariants?

We begin by remarking that the generator

$$q_\alpha^A = \sum_{i=1}^n q_{i\alpha}^A, \quad (3.17)$$

with each $q_{i\alpha}^A$ of the form (3.11), acts on the super-wave function by multiplication, just like the translation generator (2.8) in momentum representation. Therefore, exactly as in (2.7), requiring the invariance of $\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i)$ we can deduce¹³

$$p_{\alpha\dot{\alpha}} \mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = q_\alpha^A \mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = 0 \Rightarrow \mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}(p_{\alpha\dot{\alpha}}) \delta^{(8)}(q_\alpha^A) \mathcal{P}_n(\lambda, \tilde{\lambda}, \eta), \quad (3.18)$$

where $\delta^{(8)}(q_\alpha^A) = \prod_{A=1}^4 \prod_{\alpha=1,2} q_\alpha^A$ is a Grassmann delta function and \mathcal{P}_n is some polynomial in η_i^A . According to (3.18), the superamplitude (3.16) factorizes into $\delta^{(8)}(q_\alpha^A)$ of Grassmann degree 8 and another polynomial \mathcal{P}_n . As we explain in a moment, the maximal degree of \mathcal{P}_n in η is not $4n - 8$ but $4n - 16$. Moreover, since each η_i^A carries an $SU(4)$ index while \mathcal{P}_n should be a singlet, the polynomial \mathcal{P}_n can be split into a sum of $SU(4)$ singlet *homogeneous* polynomials of degree multiple of 4,

$$\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = i(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\right) \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right) \left[\mathcal{P}_n^{(0)} + \mathcal{P}_n^{(4)} + \mathcal{P}_n^{(8)} + \dots + \mathcal{P}_n^{(4n-16)}\right]. \quad (3.19)$$

¹³Consider the degenerate case where all λ_i are ‘parallel’, $\lambda_{i\alpha} = c_i \lambda_\alpha$, for some coefficients c_i . Then we have $q_\alpha^A = \lambda_\alpha \sum c_i \eta_i^A$ and the condition of q -invariance implies the presence of a factor of $\delta^{(4)}(\sum c_i \eta_i)$ of Grassmann degree 4 in the amplitude. The only case where all the λ_i are parallel is the three-point $\overline{\text{MHV}}$ amplitude (which requires complexified momenta to exist), and hence this is the unique amplitude with Grassmann degree less than 8.

We notice that $\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta)$ (3.19) serves as a generating function for all particle scattering amplitudes in the $\mathcal{N} = 4$ SYM theory. In order to extract a particular scattering amplitude out of the superamplitude (3.19), we need to expand it in terms of η_i^A and to collect terms of a given degree at each point, according to the content of the super-wave function (3.10). For example, if at point i we have a gluon G^+ , we need $(\eta_i)^0$; if the particle is a gluino Γ_A , we need $(\eta_i)^1$, etc. The highest degree $(\eta_i)^4$ is obtained if a gluon G^- occupies position i . In an amplitude with m negative-helicity gluons, appearing at points i_1, \dots, i_m , there will be a term of the form $(\eta_{i_1})^4 \dots (\eta_{i_m})^4$ of degree $4m$.

Now, the prefactor $\delta^{(8)}(q_\alpha^A)$ in (3.19), whose presence is required by supersymmetry, already has degree 8. This means that the gluon amplitudes extracted from (3.19) must have *at least two negative-helicity gluons*, which corresponds to MHV amplitudes. They are described by the first term in (3.19), with the factor $\mathcal{P}_n^{(0)}$ of degree zero in η . We see here a very simple explanation of the well-known fact [35, 36] that $\mathcal{N} = 4$ supersymmetry forbids gluon amplitudes with less than two negative-helicity gluons.

The last term in (3.19), $\mathcal{P}_n^{(4n-16)}$, multiplied by the Grassmann delta function, $\delta^{(8)}(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A)$, has degree $4(n-2)$. It contains a gluon amplitude with $n-2$ negative-helicity gluons and two positive-helicity ones. This is an $\overline{\text{MHV}}$ amplitude which can be obtained from an MHV amplitude by PCT conjugation. Having a term of higher degree in (3.19) would imply the existence of amplitudes with less than two positive-helicity gluons, which is forbidden by supersymmetry. This follows from the equivalent antiholomorphic description of the superamplitude mentioned at the end of Sect. 3.2.¹⁴ In a similar manner, the polynomial $\mathcal{P}_n^{(4k)}$ contains the n -particle non-MHV scattering amplitudes with total helicity $n-2(k+2)$ including gluon amplitudes with $(k+2)$ gluons of helicity (-1) and the remaining $(n-k-2)$ gluons with helicity $(+1)$.

The simplest amplitude in (3.19) is the MHV superamplitude

$$\mathcal{A}_n^{\text{MHV}}(\lambda, \tilde{\lambda}, \eta) = i(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\right) \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right) \mathcal{P}_n^{(0)}. \quad (3.20)$$

What can we say about the factor $\mathcal{P}_n^{(0)}$? It has Grassmann degree zero, so it is independent of the odd variables η . The two delta functions in (3.20) have no helicity, while according to (3.16) the superamplitude must have helicity $+1$ at each point. This helicity should be carried by $\mathcal{P}_n^{(0)}$. In the case of the tree-level MHV amplitude we can determine this factor by comparing it, e.g., to the gluon amplitude (2.6). As explained above, to have negative-helicity gluons at points 1 and 2 we need to extract the term $(\eta_1)^4(\eta_2)^4$ from $\delta^{(8)}(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A)$ (at the remaining points we need no η 's). In doing so, η_1^A and η_2^A form $SU(4)$ invariants, and the accompanying spinor variables λ_1 and λ_2 contract into a Lorentz invariant. The result is $(12)^4(\eta_1)^4(\eta_2)^4$, which reproduces the numerator in (2.6). Then the denominator is obtained by setting

$$\mathcal{P}_{n;0}^{(0)} = \frac{1}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad (3.21)$$

which has the required helicity $+1$ at each point. Thus, we have derived Nair's description of the n -particle MHV tree-level superamplitude

$$\mathcal{A}_{n;0}^{\text{MHV}} = \frac{i(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\right) \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}. \quad (3.22)$$

¹⁴An alternative explanation is provided by \bar{q} supersymmetry, see below.

Extracting other partial amplitudes from (3.22) one follows the same procedure as above. For instance, the mixed gluon/gluino amplitude (3.1) is obtained by collecting the terms $(\eta_1)^4(\eta_2)^1(\eta_3)^3$.

Let us come back to the supersymmetry generators (3.11). We have constructed the superamplitude (3.19) in such a way that the invariance under the first of them, q_α^A , is manifest (just like translation invariance). Let us now consider the second generator

$$\bar{q}_{A\dot{\alpha}} = \sum_{i=1}^n \tilde{\lambda}_{i\dot{\alpha}} \frac{\partial}{\partial \eta_i^A}. \quad (3.23)$$

Acting on the argument of $\delta^{(8)}(\sum_{j=1}^n \lambda_{i\alpha} \eta_j^A)$ in (3.19), it gives

$$\bar{q}_{A\dot{\alpha}} \left(\sum_{i=1}^n \lambda_{i\alpha} \eta_i^B \right) = \sum_{i=1}^n \lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}}, \quad (3.24)$$

which vanishes due to the momentum conservation delta function $\delta^{(4)}(\sum_{i=1}^n \lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}})$ in (3.19). We conclude that the MHV superamplitude (3.22) is *invariant under the full on-shell supersymmetry*. As to the other terms in (3.19), the second supersymmetry condition,

$$\bar{q}_{A\dot{\alpha}} \mathcal{P}_n^{(4k)} = 0, \quad (k = 1, \dots, n-4), \quad (3.25)$$

imposes restrictions on their η dependence. For example, the absence of the two terms in (3.19), $\mathcal{P}_n^{(4n-12)} = \mathcal{P}_n^{(4n-8)} = 0$, was explained above by the properties of the $\overline{\text{MHV}}$ amplitude. In fact, this is equivalent to requiring \bar{q} -invariance. Indeed, the generator (3.23) acts on the η 's as follows:

$$\delta_{\bar{q}} \eta_i^A = \bar{\rho}_{\dot{\alpha}}^A \tilde{\lambda}_{i\dot{\alpha}}, \quad (3.26)$$

with $\bar{\rho}_{\dot{\alpha}}^A$ being an odd antichiral parameter. This parameter has two components, which can be used to put to zero any two η 's. The remaining $(n-2)$ nonvanishing η 's form a \bar{q} -invariant of maximal degree $4(n-2)$, of which 8 is already present in the compulsory $\delta^{(8)}(\sum_{i=1}^n \lambda_{i\alpha} \eta_i^A)$. Thus, the maximal degree left for the factors $\mathcal{P}_n^{(4k)}$ in (3.19) is $4(n-4)$, as stated above. Furthermore, as we will show in Sect. 6, \bar{q} -supersymmetry imposes even stronger restrictions on the NMHV amplitude (the factor $\mathcal{P}_n^{(4)}$ in (3.19)).

As was mentioned at the end of Sect. 2.1, the tree-level MHV amplitudes have a conventional conformal symmetry [1] and, in particular, they are annihilated by the conformal boost generator $k_{\alpha\dot{\alpha}}$, Eq. (2.10). Combined together, supersymmetry and conformal symmetry lead to the $\mathcal{N} = 4$ *superconformal* symmetry of the superamplitudes (3.22). The corresponding special superconformal generators have the form:

$$s_A^\alpha = \sum_{i=1}^n \frac{\partial^2}{\partial \lambda_{i\alpha} \partial \eta_i^A}, \quad \bar{s}_{\dot{\alpha}}^A = \sum_{i=1}^n \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_{i\dot{\alpha}}}. \quad (3.27)$$

To show that s_A^α annihilates the amplitude (3.22) requires a short calculation [1]. The invariance under $\bar{s}_{\dot{\alpha}}^A$ is obvious, due to the fact that acting on (3.22) the generator $\bar{s}_{\dot{\alpha}}^A$ shifts the argument of the momentum delta function in (3.22) by the amount proportional to the argument of the Grassmann delta function. Together, these symmetries imply invariance of the superamplitude under $k_{\alpha\dot{\alpha}}$.

4 Dual superconformal symmetry of the $\mathcal{N} = 4$ superamplitudes

In this section we extend the notions of full space and dual space from Sect. 2 to chiral superspaces (subsection 4.2), starting from the on-shell superspace of Sect. 3.2. With the help of these spaces we define the action of dual superconformal symmetry in Sect. 4.3. In Sect. 4.4 we rewrite the tree-level MHV superamplitude as a dual superconformal covariant function in dual superspace. Then, we discuss the dual conformal properties of its various components and, in particular, explain why only the split-helicity amplitudes are manifestly dual conformal.

4.1 Full superspace

In the preceding section we saw that the superamplitudes have the following general form in the on-shell superspace,

$$\mathcal{A}_n = i(2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \right) \delta^{(8)} \left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A \right) \mathcal{P}_n(\lambda_i, \tilde{\lambda}_i, \eta_i). \quad (4.1)$$

The function \mathcal{P}_n depends on the variables λ_i , $\tilde{\lambda}_i$ and η_i (with $i = 1, \dots, n$) which are constrained by the two delta functions. As before, the spinor variables λ_i and $\tilde{\lambda}_i$ have to verify the momentum conservation (see (2.12)). In addition, the variables λ_i and η_i have to satisfy the relation $\sum_{i=1}^n \lambda_i^\alpha \eta_i^A = 0$, which reflects the invariance of the superamplitude under q -supersymmetry. In very much the same way as it was done in Sect. 2.2, both conditions can be trivially resolved by introducing new dual variables. Namely, we introduce dual $x_i^{\dot{\alpha}\alpha}$ coordinates to solve the momentum conservation constraint and chiral dual $\theta_i^{A\alpha}$ coordinates to solve for the supercharge conservation constraint,

$$\begin{aligned} \sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = 0 &\implies x_i^{\dot{\alpha}\alpha} - x_{i+1}^{\dot{\alpha}\alpha} = \tilde{\lambda}_1^{\dot{\alpha}} \lambda_i^\alpha, \\ \sum_{i=1}^n \lambda_i^\alpha \eta_i^A = 0 &\implies \theta_i^{A\alpha} - \theta_{i+1}^{A\alpha} = \lambda_i^\alpha \eta_i^A, \end{aligned} \quad (4.2)$$

and we impose the cyclicity conditions

$$x_{n+1} \equiv x_1, \quad \theta_{n+1} \equiv \theta_1. \quad (4.3)$$

We will call the space with coordinates $(\lambda_i, \tilde{\lambda}_i, x_i, \eta_i, \theta_i)$ the ‘full superspace’.

We can think of the relations (4.2) as defining a surface in the full superspace. Then we can interpret the function \mathcal{P}_n appearing in the amplitude (4.1) as a function on this surface. It is clear that \mathcal{P}_n can only depend on the dual x - and θ -coordinates through their differences $x_i - x_j = x_{ij}$ and $\theta_i - \theta_j = \theta_{ij}$, thus implying dual (super)translation invariance,

$$P_{\alpha\dot{\alpha}} \mathcal{P}_n = 0, \quad Q_{A\alpha} \mathcal{P}_n = 0. \quad (4.4)$$

Again, we should stress that the generators of these symmetries,

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}, \quad Q_{A\alpha} = \sum_{i=1}^n \frac{\partial}{\partial \theta_i^{A\alpha}}, \quad (4.5)$$

are not related to the usual translation generator $p_{\alpha\dot{\alpha}}$, Eq. (2.8), or supercharge q_α^A , Eq. (3.17), (an obvious difference is the type of $SU(4)$ index of the dual supercharge $Q_{\alpha A}$ as opposed to that of q_α^A). As before, $P_{\alpha\dot{\alpha}}$ generates shifts of dual x -variables, while $Q_{A\alpha}$ generates shifts of θ 's

$$\delta_Q \theta_i^{A\alpha} = \epsilon^{A\alpha}, \quad (4.6)$$

with $\epsilon^{A\alpha}$ being a constant odd parameter (a chiral Weyl spinor).

The (super)translation invariance (4.4) can be equivalently interpreted as the possibility to solve for the dual coordinates x_i and θ_i from (4.2), up to the freedom of choosing the arbitrary reference points, e.g., x_1 and θ_1 :

$$(x_i)^{\dot{\alpha}\alpha} = (x_1)^{\dot{\alpha}\alpha} - \sum_{k=1}^{i-1} \tilde{\lambda}_k^{\dot{\alpha}} \lambda_k^\alpha, \quad \theta_i^{A\alpha} = \theta_1^{A\alpha} - \sum_{k=1}^{i-1} \lambda_k^\alpha \eta_k^A. \quad (4.7)$$

In other words, the definition of the dual coordinates (4.2) is invariant under shifts of x_i and θ_i by an arbitrary constant vector and spinor, respectively. Clearly, using the dual translation and supersymmetry invariance (4.4) and the constraints (4.2), we can return to the on-shell superspace with just $\lambda_i, \tilde{\lambda}_i, \eta_i$ as coordinates.

4.2 Dual superspace

Alternatively, we can give a holomorphic description of the superamplitudes by eliminating $\tilde{\lambda}_i$ and η_i instead of x_i and θ_i . As in the bosonic case (see Sect. 2.3), we can rewrite the constraints (4.2) without using the variables $\tilde{\lambda}_i$ and η_i ,

$$(x_{i+1})^{\dot{\alpha}\alpha} \lambda_{i\alpha} = 0, \quad (\theta_{i+1})^{A\alpha} \lambda_{i\alpha} = 0. \quad (4.8)$$

These relations are equivalent to (4.2). Indeed, the general solution to (4.8) takes the form

$$(x_{i+1})^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha, \quad (\theta_{i+1})^{A\alpha} = \lambda_i^\alpha \eta_i^A, \quad (4.9)$$

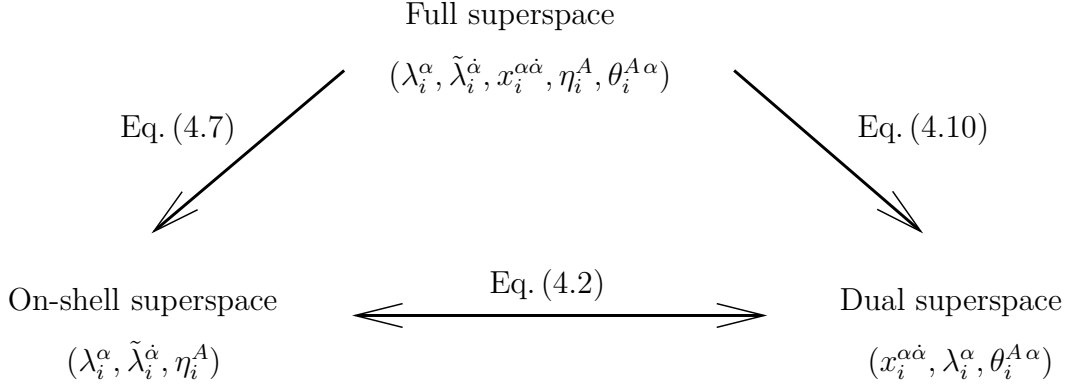
thus introducing the *secondary* variables $\tilde{\lambda}_i$ and η_i . Namely, $\tilde{\lambda}_i$ and η_i can be expressed in terms of the x_i, λ_i and θ_i by projecting both relations in (4.9) by, e.g., $\lambda_{i+1\alpha}$:

$$\tilde{\lambda}_i^{\dot{\alpha}} = \frac{(x_{i+1})^{\dot{\alpha}\alpha} \lambda_{i+1\alpha}}{\langle i+1 | i \rangle}, \quad \eta_i^A = \frac{(\theta_{i+1})^{A\alpha} \lambda_{i+1\alpha}}{\langle i+1 | i \rangle}. \quad (4.10)$$

With these relations taken into account, the function \mathcal{P}_n can now be regarded as a function of the variables x_i, λ_i, θ_i .

We call the space with coordinates $(x_i, \lambda_i, \theta_i)$, satisfying the constraints (4.8), the ‘dual superspace’. It is important to realize that this is a *chiral* superspace, we only use chiral spinors λ_i^α and $\theta_i^{A\alpha}$, but not their antichiral complex conjugates. The relations between the on-shell, full

and dual superspaces are summarized in the following diagram:



4.3 Dual superconformal symmetry

In this subsection, we extend the previous analysis (see Sect. 2.4) of dual conformal properties of the bosonic coordinates $x, \lambda, \tilde{\lambda}$ to the fermionic coordinates θ, η . Starting from the known transformation properties of the ‘odd’ dual coordinates $\theta^{A\alpha}$ under inversion, we derive those of the on-shell variables η^A . In this way we complete dual conformal symmetry $SO(2, 4)$ to the superconformal symmetry $SU(2, 2|4)$.

4.3.1 Dual Poincaré supersymmetry

In the dual superspace with coordinates $(x_i, \lambda_i, \theta_i)$ we introduce the generators

$$Q_{A\alpha} = \sum_{i=1}^n \frac{\partial}{\partial \theta_i^{A\alpha}}, \quad \bar{Q}_{\dot{\alpha}}^A = \sum_{i=1}^n \theta_i^{A\alpha} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}, \quad P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}, \quad (4.11)$$

satisfying the $\mathcal{N} = 4$ Poincaré supersymmetry algebra

$$\{Q_{A\alpha}, \bar{Q}_{\dot{\alpha}}^B\} = \delta_A^B P_{\alpha\dot{\alpha}}. \quad (4.12)$$

The generator $\bar{Q}_{\dot{\alpha}}^A$ has an induced action on the on-shell superspace variables η , which follows from (4.10):

$$\bar{Q}_{\dot{\alpha}}^A = \sum_{i=1}^n \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}. \quad (4.13)$$

Both forms of $\bar{Q}_{\dot{\alpha}}^A$ can be obtained from its representation in the full superspace (B.8) by restricting to the dual superspace or on-shell superspace, respectively. Note that the action of $\bar{Q}_{\dot{\alpha}}^A$ on the on-shell superspace (4.13) is identical to the ordinary superconformal generator $\bar{s}_{\dot{\alpha}}^A$, Eq. (3.27), acting in the on-shell superspace. We could say that half of the dual Poincaré supersymmetry, $\bar{Q}_{\dot{\alpha}}^A$, is induced by the ordinary superconformal symmetry $\bar{s}_{\dot{\alpha}}^A$. In Sect. 4.3.2 we will extend this dual Poincaré supersymmetry to the full $\mathcal{N} = 4$ superconformal symmetry $SU(2, 2|4)$.

Now we recall the discussion of the holomorphic approach to the (super)amplitudes from Sect. 2.3. In the bosonic case we chose to describe the amplitude $\mathcal{A}(x, \lambda)$ in terms of the dual

space coordinates (x_i, λ_i) and not to consider their complex conjugates. In the supersymmetric case the analogous choice is that of the *chiral* dual superspace (x, θ, λ) . What motivates this choice? In Sect. 4.2 we saw that from the chiral dual superspace we can deduce the existence of the on-shell variables η . In Sect. 3.2 we have shown that the complete, PCT self-conjugate on-shell gluon supermultiplet can be described in a holomorphic way, in terms of η^A only. It is precisely this special property of the $\mathcal{N} = 4$ SYM theory which makes it possible to define purely chiral superamplitudes.

Thus, we can say that the choice of the chiral dual superspace is determined by the holomorphic description of the on-shell gluon multiplet. Further, the chiral Grassmann coordinates θ_α^A have twice the number of degrees of freedom of the on-shell variables η^A , which justifies the fermionic defining constraint in (4.9). Without the auxiliary spinor variables, in order to ‘halve’ the chiral spinor θ_α^A , we would have to explicitly break Lorentz symmetry. As discussed earlier in Sect. 3.2, the role of the auxiliary spinor variables λ is to make these light-cone projections manifestly covariant.

Of course, we could make the equivalent choice of an antichiral dual superspace, corresponding to the antiholomorphic description of the gluon multiplet in terms of $\bar{\eta}$ (see (3.14)). The important point is that the specific nature of the $\mathcal{N} = 4$ gluon multiplet allows us to use either the one or the other description, and does not oblige us to mix them.

4.3.2 Chiral realization of the dual $SU(2, 2|4)$

As in Sect. 2.3, we treat the coordinates of the chiral dual superspace (x, λ, θ) as our ‘primary’ variables. Let us first discuss their superconformal properties and, then, derive the transformation rules for the ‘secondary’ on-shell superspace variables η .

To begin with, we need to supplement the already known conformal inversion rules for x , Eq. (2.23), and for λ , Eq. (2.31), with the standard rule for the odd superspace coordinates θ [37],

$$I [\theta_i^{A\alpha}] = (x_i^{-1})^{\dot{\alpha}\beta} \theta_{i\beta}^A, \quad I [\theta_i^A] = \theta_i^{A\beta} (x_i^{-1})_{\beta\dot{\alpha}}. \quad (4.14)$$

It is easy to see that the defining constraints (4.9) transform covariantly under these transformations, so the chiral dual superspace is closed under conformal inversion.

The combination of the dual supersymmetry transformation (4.6) with inversion implies another continuous symmetry with generator $\bar{S}_A^{\dot{\alpha}} = I Q_{A\alpha} I$, in close analogy with the conformal boosts $K^\mu = I P^\mu I$. Its action on the odd dual coordinates θ is easy to work out. After the inversion θ_i becomes $\theta_i x_i^{-1}$. Then, after a dual supersymmetry transformation we get $(\theta_i + \epsilon) x_i^{-1}$. Finally, the second inversion brings us to $\theta_i + \bar{\rho} x_i$ with $\bar{\rho}_\alpha = I[\epsilon_\alpha]$ (as usual, inversion changes the chirality of the spinors, including the transformation parameters) leading to

$$\delta_{\bar{S}} \theta_i^{\alpha A} = \bar{\rho}_\alpha^A x_i^{\dot{\alpha}\alpha}. \quad (4.15)$$

The generator of this transformation in the dual superspace only acts on θ_i and leaves the other variables x_i and λ_i intact

$$\bar{S}_A^{\dot{\alpha}} = \sum_{i=1}^n x_i^{\dot{\alpha}\alpha} \frac{\partial}{\partial \theta_i^{A\alpha}}. \quad (4.16)$$

Commuting \bar{S} with the translation generator (4.5), we obtain

$$[\bar{S}_A^{\dot{\alpha}}, P_{\beta\dot{\beta}}] = \delta_{\beta}^{\dot{\alpha}} Q_{A\beta}. \quad (4.17)$$

Next, applying inversion to both sides of this commutator, using $\bar{S} = IQI$, $K = IPI$ and $I^2 = \mathbb{I}$, we obtain

$$[Q_{\alpha A}, K_{\beta \dot{\beta}}] = \epsilon_{\alpha\beta} \bar{S}_{A \dot{\beta}}. \quad (4.18)$$

We identify the generators P, K, Q, \bar{S} as part of the $\mathcal{N} = 4$ superconformal algebra $su(2, 2|4)$. The explicit form of the generators of this algebra and their commutation relations can be found in Appendix B.

The reason why we have $su(2, 2|4)$ and not $psu(2, 2|4)$ is that the algebra involves a central charge. To see this, consider the anticommutators $\{Q, S\}$ and $\{\bar{Q}, \bar{S}\}$ from Eq. (B.3). The Lorentz (M) and $SU(4)$ (R) generators annihilate the scalar and singlet amplitude, while the action of the dilatation operator D and the central charge C on the tree-level superamplitude (3.19) and (3.22) is given by

$$D \mathcal{A}_{n;0}(\lambda, \tilde{\lambda}, \eta) = C \mathcal{A}_{n;0}(\lambda, \tilde{\lambda}, \eta) = -n \mathcal{A}_{n;0}(\lambda, \tilde{\lambda}, \eta). \quad (4.19)$$

Examining the explicit realizations of these two generators, Eqs. (B.12) and (B.13), respectively, we see that λ and $\tilde{\lambda}$ have the same dilatation weight ($-1/2$), while they have opposite central charges, $(-1/2)$ for λ and $(+1/2)$ for $\tilde{\lambda}$. This suggests to identify the central charge with helicity.

4.3.3 Induced action on the on-shell odd variables

The fact that the η 's are determined by the θ 's and λ 's (recall (4.10)) implies that their conformal properties follow from those of θ and λ . Let us begin by making an inversion in the second equation in (4.9), with the help of (4.14) and (2.31),

$$(\theta_i^A x_i^{-1})_{\dot{\alpha}} - (\theta_{i+1}^A x_{i+1}^{-1})_{\dot{\alpha}} = (\lambda_i x_i^{-1})_{\dot{\alpha}} I[\eta_i^A]. \quad (4.20)$$

Then we multiply this equation by $(x_i)^{\dot{\alpha}\alpha}$ from the right, replace θ_{i+1}^A by $\theta_i^A - \lambda_i \eta_i^A$ using (4.8) and apply (2.20) to arrive at

$$\lambda_i^\alpha \frac{x_i^2}{x_{i+1}^2} \left(\eta_i^A - \theta_i^A x_i^{-1} \tilde{\lambda}_i \right) = \lambda_i^\alpha I[\eta_i^A]. \quad (4.21)$$

Since this equation should hold for any λ_i^α , we deduce

$$I[\eta_i^A] = \frac{x_i^2}{x_{i+1}^2} \left(\eta_i^A - \theta_i^A x_i^{-1} \tilde{\lambda}_i \right), \quad (4.22)$$

where the contraction of spinor indices is tacitly implied. Repeating the inversion twice, we obtain $I^2[\eta_i^A] = \eta_i^A$, as expected.

It is important to realize that, contrary to θ -variables, the transformation of η in (4.22) is *not homogeneous* (η transforms through itself and through θ). In addition, the relation (4.22) explicitly involves the antichiral spinor variable $\tilde{\lambda}$, which takes us out of the holomorphic description. This shows that the η -variables are not well suited for the discussion of the dual conformal properties of the superamplitude. We will come back to this important point in Sect. 4.4.

The infinitesimal dual superconformal transformation (4.15) of θ induces that of η ,

$$\delta_{\bar{S}} \eta_i^A = \bar{\rho}_{\dot{\alpha}}^A \tilde{\lambda}_i^{\dot{\alpha}} \quad (4.23)$$

and the corresponding generator $\bar{S}_A^{\dot{\alpha}}$ in the on-shell superspace is ¹⁵

$$\bar{S}_A^{\dot{\alpha}} = \sum_{i=1}^n \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^A} \quad (4.24)$$

We remark that the transformation (4.23) is identical to (3.26), and its generator (4.24) coincides with the supersymmetry generator $\bar{q}_{A\dot{\alpha}}$ (3.11) acting in the on-shell superspace. We can reverse the argument and say that the light-cone supercharge $\bar{q}_{A\dot{\alpha}}$, initially acting on the on-shell superspace variables η , induces the \bar{S} transformations of the dual superspace coordinates θ through the relation (4.2).

4.3.4 Transformation properties of the delta function in the superamplitude

In Sect. 3.3 we have shown that the superamplitude (3.19) contains a prefactor made of two delta functions, bosonic and fermionic. The dual space interpretation of the bosonic delta function was given in Sect. 2.5: we first broke the n -point cycle, $x_{n+1} \neq x_1$, and then used the delta function to impose back the identification $x_{n+1} = x_1$. In complete analogy, we first relax the cyclicity condition (4.3) and then replace the product of two delta functions in (3.19) by

$$\delta^{(4)}(x_1 - x_{n+1}) \delta^{(8)}(\theta_1^{Aa} - \theta_{n+1}^{A\alpha}). \quad (4.25)$$

This product imposes the condition (4.3). The advantage of this reformulation, from the point of view of dual conformal symmetry, is that the covariance of (4.25) under inversion (4.14) is manifest, assuming that θ_{n+1}^A transforms through x_{n+1} . As in the case of the bosonic delta function, this creates some extra conformal weight at the preferred point 1. We will come back to this point in Sect. 4.4.

The invariance of the Grassmann delta function in (4.25) under the dual Q supersymmetry (4.5) is obvious. To show the invariance under the dual special conformal supersymmetry (4.16) (which is equivalent to the light-cone supersymmetry (3.24)), we again need the help of the bosonic delta function.

Finally, an interesting question is what is the role of the other half of the ‘odd’ $SU(2, 2|4)$ generators \bar{Q} , Eqs. (4.11) and (4.13), and S , Eq. (B.15). Applying \bar{Q} to the argument of bosonic delta function in (4.25), we obtain $\bar{Q}^A(x_1 - x_{n+1}) = \theta_1^A - \theta_{n+1}^A$, which vanishes due to the Grassmann delta function in (4.25). Hence, the MHV tree-level superamplitude (3.22) is invariant under the full dual $\mathcal{N} = 4$ Poincaré supersymmetry. Combining this with dual conformal symmetry, we can say that it is covariant under the full dual $SU(2, 2|4)$. However, as soon as we turn on the perturbative corrections, which involve non-trivial dependence on x , the role of \bar{Q} (and consequently of S) becomes less clear. One point we can make is that, unlike \bar{S} , \bar{Q} cannot remain an exact symmetry of the amplitude. Indeed, the two symmetries imply the compatibility condition (see (B.3))

$$\bar{Q}\mathcal{A}_n = \bar{S}\mathcal{A}_n = 0 \quad \implies \quad D\mathcal{A}_n = C\mathcal{A}_n. \quad (4.26)$$

This relation holds at tree level, but it does not survive loop corrections since the dilatation symmetry becomes anomalous due to the presence of infrared divergences, while the central charge C still measures the helicity of the superamplitude and, therefore, is protected. We hope to come back to this issue in the future.

¹⁵The representations of $\bar{S}_A^{\dot{\alpha}}$ in the dual superspace (4.16) and in the on-shell superspace (4.24) can be obtained from the representation on the full superspace (B.15).

4.4 Dual superconformal properties of the tree-level MHV superamplitude

The main result of the preceding subsections was the introduction of dual conformal symmetry (inversion rules (2.31) and (4.14)). We have shown that the denominator in the MHV superamplitude (3.22) is covariant under these transformations. The superamplitude also involves two delta functions whose origin is (super)translation invariance in the on-shell superspace $(\lambda, \tilde{\lambda}, \eta)$. As explained above, their dual conformal properties become manifest if we first break the cyclicity of the amplitude by introducing an extra point in dual superspace, $x_{n+1} \neq x_1$, $\theta_{n+1} \neq \theta_1$, and then use the delta functions (4.25) to identify the end points. However, this creates extra conformal weight at the breaking point (x_1, θ_1) , which seems unnatural for a cyclicly symmetric amplitude.

Fortunately, in the special case of $\mathcal{N} = 4$ dual supersymmetry the product of two delta functions (4.25) has *vanishing conformal weight* at point $1 \equiv n+1$. Indeed, under inversion the bosonic delta function transforms with a weight opposite to that of the space measure, $\int d^4x \rightarrow \int d^4x x^{-8}$, thus $\delta^{(4)}(x_1 - x_{n+1}) \rightarrow x_1^8 \delta^{(4)}(x_1 - x_{n+1})$. At the same time, since $\theta_1 - \theta_{n+1} \rightarrow x_1^{-1} \theta_1 - x_{n+1}^{-1} \theta_{n+1}$ under inversion (4.14), we have $\delta^{(8)}(\theta_1 - \theta_{n+1}) \rightarrow x_1^{-8} \delta^{(8)}(\theta_1 - \theta_{n+1})$ due to $x_1 = x_{n+1}$, so that the product (4.25) remains invariant.

This shows that we could have chosen to break the cycle at any point p , without affecting the conformal properties of the amplitude. To restore the cyclic symmetry we can sum over all such choices. This leads the following manifestly dual superconformal covariant expression for the tree-level MHV superamplitude

$$\mathcal{A}_{n;0}^{\text{MHV}} = \frac{1}{n} \sum_{p=1}^n \frac{\delta^{(4)}(x_p - x_{n+p}) \delta^{(8)}(\theta_p - \theta_{n+p})}{\langle 1\,2 \rangle \langle 2\,3 \rangle \dots \langle n\,1 \rangle}, \quad (4.27)$$

where x - and θ -variables satisfy the defining relations (4.2) with $x_{n+p} \neq x_p$ and $\theta_{n+p} \neq \theta_p$. The superamplitude (4.27) is obviously Lorentz and $SU(4)$ invariant. It also has helicity weights $+1$ at each point, so that the total helicity equals n in an agreement with (4.19). Moreover, it transforms covariantly under inversion and has equal conformal weights $+1$ at each point

$$I[\mathcal{A}_{n;0}^{\text{MHV}}] = (x_1^2 x_2^2 \dots x_n^2) \mathcal{A}_{n;0}^{\text{MHV}}. \quad (4.28)$$

However, the representation for the MHV superamplitude (4.27) is not suitable for the practical purpose of extracting various components of the superamplitude, e.g., gluon amplitudes like (2.6), etc. To this end, it is necessary to go back to the original form (3.22), where we explicitly see the on-shell superspace variables η .

In Sects. 2.5 and 2.6 we have shown that the special components of the superamplitude, the split-helicity amplitudes, have covariant dual conformal transformations, while the rest do not. What is the reason that the manifest dual conformal covariance of the superamplitude (4.28) is lost when going down to its components? The answer can be found in the *inhomogeneous* transformation of the η 's, Eq. (4.22). Indeed, let us use Q supersymmetry (4.6) to set, e.g., $\theta_1 = 0$ (this choice is compatible with dual conformal invariance, see (4.14)). Then, using (4.7) we can rewrite (4.22) as

$$I[\eta_i] = \frac{x_i^2}{x_{i+1}^2} \left(\eta_i + \sum_{k=1}^{i-1} \eta_k \langle k | x_i^{-1} | i \rangle \right). \quad (4.29)$$

We see that, due to the presence of inhomogeneous term in this relation, the different η terms in the expansion of the superamplitude can mix with each other under inversion, yielding complicated inhomogeneous transformations for their coefficients (partial scattering amplitudes).

Let us give an example which illustrates this effect. Consider the MHV tree-level superamplitude (4.27) and take the term $p = 1$. Its components originate from the expansion of the Grassmann delta function in (4.27) or (3.22):

$$\delta^{(8)}(\theta_1 - \theta_{n+1}) = \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right) = \sum_{1 \leq i < j \leq n} \langle i j \rangle^4 (\eta_i^4)^4 (\eta_j^4)^4 + \dots, \quad (4.30)$$

where only purely gluon components are shown. We already know the behavior of this delta function under inversion,

$$I\left[\delta^{(8)}(\theta_1 - \theta_{n+1})\right] = x_1^{-8} \delta^{(2)}(\theta_1 - \theta_{n+1}) \quad (4.31)$$

(taking into account the bosonic $\delta^{(4)}(x_1 - x_{n+1})$). What can we say about the conformal properties of its components? In general they are not simple, because of the inhomogeneous term in (4.22). The exception are the split-helicity amplitudes, for which, e.g., the two negative-helicity gluons appear at adjacent points i and $i + 1$. Consider, e.g., the term $\langle n - 1 \ n \rangle^4 (\eta_{n-1})^4 (\eta_n)^4$ in the right-hand side of (4.30). From (4.29) it is clear that only this term is not affected by the inhomogeneous transformations, giving (recall (2.33))

$$\begin{aligned} I[\langle n - 1 \ n \rangle^4 (\eta_{n-1})^4 (\eta_n)^4] &= \frac{\langle n - 1 \ n \rangle^4}{x_{n-1}^8} \frac{x_{n-1}^8}{x_1^8} (\eta_{n-1})^4 (\eta_n)^4 + \dots \\ &= x_1^{-8} \langle n - 1 \ n \rangle^4 (\eta_{n-1})^4 (\eta_n)^4 + \dots, \end{aligned} \quad (4.32)$$

(here \dots denotes terms of different types), in accord with (4.31). To show the covariance of the other split helicity terms $(\eta_i)^4 (\eta_{i+1})^4$ in (4.30), we need to make a different choice for the ‘starting point’ $\theta_1 = 0$ of the cycle.

The same example shows what happens to other amplitudes, which are not of the split-helicity type. Take, for instance, the term $(\eta_{n-2})^4 (\eta_n)^4$ in (4.30). It mixes under inversion with similar term $x_1^{-8} \langle n - 2 | x_{n-1}^{-1} x_{n-1} \ n | n \rangle^4 (\eta_{n-2})^4 (\eta_n)^4$ coming from inhomogeneous transformation of $(\eta_{n-1})^4 (\eta_n)^4$. This explains why the MHV gluon amplitude $A_n(1^+ \dots (n-2)^- (n-1)^+ n^-)$ does not have a homogeneous dual conformal transformation.

4.5 Conventional and dual superconformal generators

Just as in the purely bosonic case we can deduce the form of all generators of dual superconformal transformations by working in the full superspace with coordinates $x_i, \theta_i, \lambda_i, \tilde{\lambda}_i, \eta_i$. In this superspace the superamplitude is supported on a surface defined by the constraints

$$x_i^{\dot{\alpha}\alpha} - x_{i+1}^{\dot{\alpha}\alpha} - \tilde{\lambda}_i^{\dot{\alpha}} \lambda_i^\alpha = 0, \quad \theta_i^{A\alpha} - \theta_{i+1}^{A\alpha} - \lambda_i^\alpha \eta_i^A = 0. \quad (4.33)$$

Following the same logic as we used for the bosonic constraints in Sect. 2.7, we can extend the dual conformal generator acting on (x, θ) to the surface (4.33) in the full superspace. The resulting generator,

$$K^{\dot{\alpha}\alpha} = \sum_{i=1}^n \left[x_i^{\dot{\beta}\alpha} x_i^{\dot{\alpha}\beta} \frac{\partial}{\partial x_i^{\dot{\beta}\beta}} + x_i^{\dot{\alpha}\beta} \theta_i^{B\alpha} \frac{\partial}{\partial \theta_i^{\beta B}} + x_i^{\dot{\alpha}\beta} \lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\beta} + x_{i+1}^{\dot{\beta}\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\alpha}} \theta_{i+1}^{B\alpha} \frac{\partial}{\partial \eta_i^B} \right], \quad (4.34)$$

defines the dual conformal transformation of all variables, $x_i, \theta_i, \lambda_i, \tilde{\lambda}_i, \eta_i$.

We can obtain the form of $K^{\dot{\alpha}\alpha}$ in the on-shell superspace by ignoring the first two terms in (4.34). As in the bosonic case, the conformal transformation introduces a dependence on the variables x_i, θ_i which do not live in the on-shell superspace. To obtain the generators acting in the dual superspace we can ignore the final two terms in (4.34). In the same way we can find all generators of the dual superconformal algebra $u(2, 2|4)$. These are given in Appendix B.

The following picture summarizes the relationship between the two superconformal algebras, the conventional one (acting in the configuration superspace of the particles) and the dual one (acting in the dual superspace and shown in capital letters):

$$\begin{array}{ccccc}
& p & & K & \\
q & & \bar{q} = \bar{S} & & S \\
s & & \bar{s} = \bar{Q} & & Q \\
& k & & P &
\end{array}$$

On the left-hand (conventional) side we have the generators p and q which are trivially realized on the superamplitude (they vanish due to the delta functions). Further, the generators k and s are realized in terms of second-order differential operators (see (2.10) and (3.27)), so the implementation of these symmetries is not straightforward. On the dual side, P and Q act as (super)translations leading to the elimination of one point, e.g., (x_1, θ_1) in the dual superspace. The generators K and S correspond to exact symmetries of the tree-level superamplitude, but they become anomalous when the loop corrections are switched on. The overlap between the two superalgebras is over the generators $\bar{q} = \bar{S}$ and $\bar{s} = \bar{Q}$. The former remain exact symmetries of the full superamplitude, while the latter are again subject to anomalies due to infrared divergences.

5 The complete $n = 6$ NMHV superamplitude

In this section, we shall construct the one-loop NMHV superamplitude for $n = 6$ and compare it with the one-loop expressions for six-gluon NMHV amplitudes computed in [19]. We would like to mention that another approach to constructing the $n = 6$ superamplitude based on the unitary cuts was proposed in Ref. [39].

We recall that the superamplitude $\mathcal{A}_6^{\text{NMHV}}$ is a generating function of the scattering amplitudes (more precisely, planar color-ordered partial amplitudes) of scalars, gluinos and gluons with the total helicity $n - 6 = 0$. These amplitudes can be read as the coefficients in expansion of $\mathcal{A}_6^{\text{NMHV}}$ in powers of η 's. In particular, the six-gluon amplitudes are accompanied by the $SU(4)$ singlets $(\eta_i)^4(\eta_j)^4(\eta_k)^4$ (with $(\eta_i)^4 \equiv \frac{1}{4!}\epsilon_{ABCD}\eta_i^A\eta_i^B\eta_i^C\eta_i^D$)

$$\mathcal{A}_6^{\text{NMHV}} = \sum_{1 \leq i < j < k \leq 6} (\eta_i)^4(\eta_j)^4(\eta_k)^4 A_6(i^- j^- k^-) + \dots \quad (5.1)$$

where $A_6^{\text{NMHV}}(i^- j^- k^-)$ stands for the NMHV six-gluon scattering amplitude with i^- , j^- and k^- denoting gluons with helicity (-1) and remaining gluons carry helicity $(+1)$. Also, the ellipses represent the scattering amplitudes of scalars and gluinos. The scattering amplitudes $A_6(i^- j^- k^-)$ are invariant under cyclic shifts, $i \rightarrow i+1$ and flips, $i \rightarrow 7-i$, of six gluons. As a result, for $n=6$ there are only three nontrivial NMHV amplitudes [40] that can be denoted according to the ordering of the gluon helicities as A^{+++-} , A^{+-+-} and A^{+-+-} , so that $A^{+++-} \equiv A(1^+ 2^+ 3^+ 4^- 5^- 6^-)$ and so on.

5.1 Tree level

As we argued in Sect. 3.3, all six-particle amplitudes can be combined into a single superamplitude \mathcal{A}_6 given by (3.19). The $n=6$ NMHV amplitude corresponds to the term involving $\mathcal{P}_6^{(4)}$. At tree level, we take into account (3.22) and obtain

$$\mathcal{A}_{6;0}^{\text{NMHV}} = \mathcal{A}_{6;0}^{\text{MHV}} \mathcal{P}_{6;0}^{(4)}(\lambda_i, \tilde{\lambda}_i, \eta_i), \quad (5.2)$$

where $\mathcal{A}_{6;0}^{\text{MHV}}$ is the tree-level MHV amplitude

$$\mathcal{A}_{6;0}^{\text{MHV}} = i(2\pi)^4 \frac{\delta^{(4)}(\sum_{i=1}^6 \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}) \delta^{(8)}(\sum_{i=1}^6 \lambda_i^\alpha \eta_i^A)}{\langle 12 \rangle \langle 23 \rangle \dots \langle 61 \rangle}, \quad (5.3)$$

and $\mathcal{P}_{6;0}^{(4)}(\lambda_i, \tilde{\lambda}_i, \eta_i)$ is a homogenous polynomial in η 's of degree 4.

We expect that $\mathcal{A}_{6;0}^{\text{NMHV}}$ should have the same transformation properties with respect to dual superconformal transformations as $\mathcal{A}_{6;0}^{\text{MHV}}$ and, therefore, $\mathcal{P}_{6;0}^{(4)}(\lambda_i, \tilde{\lambda}_i, \eta_i)$ should be superinvariant. We shall construct such superinvariants for arbitrary n in Sect. 6.2. To simplify the presentation, we give here the resulting expression for $\mathcal{P}_{6;0}^{(4)}(\lambda_i, \tilde{\lambda}_i, \eta_i)$ and refer interested reader to Sect. 6.2 for more details. The superinvariant $\mathcal{P}_{6;0}^{(4)}(\lambda_i, \tilde{\lambda}_i, \eta_i)$ has the factorized form

$$\mathcal{P}_{6;0}^{(4)}(\lambda_i, \tilde{\lambda}_i, \eta_i) = \frac{1}{6} \sum_{p,q,r=1}^6 c_{pqr} \delta^{(4)}(\Xi_{pqr}), \quad (5.4)$$

where sum runs over all possible $n=6$ superinvariants labeled by $p \neq q \neq r$. The Grassmann valued $\Xi_{pqr}^A(\lambda_i, \tilde{\lambda}_i, \eta_i)$ are linear in η and have the form

$$\Xi_{pqr}^A = -\langle p | \left[x_{pq} x_{qr} \sum_{i=p}^{r-1} |i\rangle \eta_i^A + x_{pr} x_{rq} \sum_{i=p}^{q-1} |i\rangle \eta_i^A \right], \quad (5.5)$$

with all indices subject to the periodicity condition $i \equiv i+6$. The coefficients $c_{pqr}(\lambda_i, \tilde{\lambda}_i)$ do not depend on η 's and will be determined shortly by matching (5.2) and (5.4) with the known result for six-gluon NMHV amplitude. It is straightforward to verify Ξ_{pqr}^A transforms covariantly under dual superconformal transformations (see (6.7) below). For $\mathcal{P}_{6;0}^{(4)}(\lambda_i, \tilde{\lambda}_i, \eta_i)$ to be invariant under these transformations, the coefficients c_{pqr} should also transform covariantly in such a way that the dual conformal weights of c_{pqr} and $\delta^{(4)}(\Xi_{pqr})$ should compensate each other. We shall verify this property by explicit calculation in Sect. 6.2.

Let us examine the triple sum on the right-hand side of (5.4) and separate the contribution with $p=1$. The remaining terms with $2 \leq p \leq 6$ can be obtained from the one for $p=1$

by applying cyclic shifts of the indices. As follows from the definition (5.5), $\Xi_{1pq} = \Xi_{1qp}$ and we can impose the condition $p \leq q$. In addition, making use of the on-shell condition (2.20), $x_{i,i+1}|i\rangle = \langle i|x_{i,i+1} = 0$, one can verify that all Ξ_{1pq} vanish except Ξ_{135} , Ξ_{136} and Ξ_{146} . Then, examining the explicit expressions (5.5) for these quantities we find that they are related to each other through cyclic shifts of the indices

$$\begin{aligned}\Xi_{136} &= \frac{\langle 12 \rangle}{\langle 23 \rangle} \Xi_{362} = \frac{\langle 12 \rangle}{\langle 23 \rangle} \mathbb{P}^2 \Xi_{146}, \\ \Xi_{135} &= \frac{\langle 12 \rangle}{\langle 23 \rangle} \Xi_{352} = \frac{\langle 12 \rangle}{\langle 23 \rangle} \mathbb{P}^2 \Xi_{136} = \frac{\langle 12 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 45 \rangle} \mathbb{P}^4 \Xi_{146},\end{aligned}\tag{5.6}$$

where \mathbb{P} performs the cyclic shift of indices $i \rightarrow i+1$ modulo the periodicity condition $i \equiv i+6$. These relations allow us to eliminate Ξ_{136} and Ξ_{135} as well as their cyclic images from the triple sum in (5.2) and to rewrite $\mathcal{A}_{6;0}^{\text{NMHV}}$ as

$$\mathcal{A}_{6;0}^{\text{NMHV}} = \mathcal{A}_{6;0}^{\text{MHV}} [\tilde{c}_{146} \delta^{(4)}(\Xi_{146}) + (\text{cyclic})],\tag{5.7}$$

where the expression inside the square brackets is invariant under cyclic shift of indices and \tilde{c}_{146} is given by a linear combination of c_{146} and cyclicly shifted c_{135} and c_{136}

$$\tilde{c}_{146} = \frac{1}{6} \left[c_{146} + c_{514} \left(\frac{\langle 56 \rangle}{\langle 61 \rangle} \right)^4 + c_{351} \left(\frac{\langle 56 \rangle \langle 34 \rangle}{\langle 61 \rangle \langle 45 \rangle} \right)^4 \right].\tag{5.8}$$

The relation (5.7) defines the $n=6$ NMHV superamplitude at tree level and it involves only one unknown function \tilde{c}_{146} .

The function \tilde{c}_{146} can be determined by comparing scattering amplitudes of gluons, gluinos and scalars generated by (5.7) with the known expressions. To perform the comparison, we will need the expression for Ξ_{146} in terms of λ_i , $\bar{\lambda}_i$ and η_i . Using (5.5) we find after some algebra

$$\Xi_{146}^A = \langle 61 \rangle \langle 45 \rangle (\eta_4^A [56] + \eta_5^A [64] + \eta_6^A [45]).\tag{5.9}$$

Based on our analysis we expect that the expression for \tilde{c}_{146} has to ensure the dual conformal invariance of the superamplitude (5.7). In particular, to compensate the conformal weight of Ξ_{146} (see Eq. (6.7) below) it should transform under inversions in the dual space as

$$I[\tilde{c}_{146}] = \tilde{c}_{146} (x_1^2 x_4^2 x_6^2)^4,\tag{5.10}$$

so that $I[\tilde{c}_{146} \delta^{(4)}(\Xi_{146})] = \tilde{c}_{146} \delta^{(4)}(\Xi_{146})$.

5.2 Generalization to all loops

It is straightforward to generalize the relation (5.7) beyond tree level

$$\mathcal{A}_6^{\text{NMHV}} = \mathcal{A}_6^{\text{MHV}} [\tilde{c}_{146} \delta^{(4)}(\Xi_{146}) (1 + a V_{146}) + (\text{cyclic})] + O(a^2),\tag{5.11}$$

where $\mathcal{A}_6^{\text{NMHV}}$ and $\mathcal{A}_6^{\text{MHV}}$ stand for the all-loop superamplitudes, \tilde{c}_{146} is defined in (5.8) and V_{146} is a scalar function of x_i . Note that the form of Ξ_{pqr} is fixed by dual superconformal symmetry and, therefore, Ξ_{pqr} is protected from perturbative corrections.

Writing down (5.11) we have tacitly assumed that the expression inside square brackets in the right-hand side of (5.11), to which we shall refer as the ‘ratio’ of the superamplitudes $\mathcal{A}_6^{\text{NMHV}}$ and $\mathcal{A}_6^{\text{MHV}}$, possesses the dual conformal invariance beyond tree level. This property is extremely nontrivial given the fact that the dual conformal invariance of the MHV amplitude $\mathcal{A}_6^{\text{MHV}}$ is known to be broken already at one-loop level. The reason for this is that loop corrections to the amplitude contain infrared divergences which are regularized within dimensional regularization by evaluating the relevant Feynman integral in $D = 4 - 2\epsilon$ dimensions. This immediately breaks dual conformality of the amplitude and induces an anomalous contribution to the conformal Ward identities. The phenomenon is rather general and it applies to the amplitude $\mathcal{A}_6^{\text{NMHV}}$ which also has infrared divergences to any loop order. Note that the infrared divergences are not sensitive to the helicities of external particles. They have the same universal form for MHV and NMHV amplitudes and, therefore, they cancel in the ratio R_6^{NMHV} of the superamplitudes defined as

$$\mathcal{A}_6^{\text{NMHV}} = \mathcal{A}_6^{\text{MHV}} [R_6^{\text{NMHV}} + O(\epsilon)] . \quad (5.12)$$

Contrary to the superamplitudes $\mathcal{A}_6^{\text{NMHV}}$ and $\mathcal{A}_6^{\text{MHV}}$, the ratio function R_6^{NMHV} is infrared finite and, therefore, it is well-defined in $D = 4$ dimension. This suggests that R_6^{NMHV} should possess dual superconformal invariance to all loops. If so, then comparing (5.11) and (5.12), we conclude that the ratio function

$$R_6^{\text{NMHV}} = \tilde{c}_{146} [1 + aV_{146}] \delta^{(4)}(\Xi_{146}) + (\text{cyclic}) + O(a^2) \quad (5.13)$$

should be invariant under dual superconformal transformations and, as a consequence, R_6^{NMHV} has to satisfy a (nonanomalous) conformal Ward identity

$$K^{a\dot{a}} R_6^{\text{NMHV}} = D R_6^{\text{NMHV}} = 0 \quad (5.14)$$

with the conformal boost operator $K^{a\dot{a}}$ and the dilatation operator D defined in (B.16) and (B.12), respectively. As we will see in a moment, this is indeed the case, at least to one loop.

To determine the function \tilde{c}_{146} we shall apply (5.1) to extract from the $n = 6$ NMHV amplitude (5.11) the expressions for the six-gluon NMHV scattering amplitudes $A_6(i^- j^- k^-)$ and compare them with the known one-loop expressions [19]. To make use of (5.11) we have to specify the perturbative corrections to the MHV superamplitude $\mathcal{A}_6^{\text{MHV}}$. The duality relation between the MHV amplitudes and light-like Wilson loops allows us to write ¹⁶

$$\mathcal{A}_6^{(\text{MHV})} = \mathcal{A}_{6,0}^{(\text{MHV})} M_6^{(\text{MHV})} , \quad \ln M_6^{(\text{MHV})} = \ln W_6 + O(\epsilon, 1/N^2) \quad (5.15)$$

where the tree level $n = 6$ MHV superamplitude $\mathcal{A}_{6,0}^{\text{MHV}}$ is given by (3.22) and W_6 is the vacuum expectation value of the light-like Wilson loop evaluated over a hexagonal contour with vertices located at the points x_i^μ (with $i = 1, \dots, 6$) which are the dual coordinates related to the gluon momenta $x_i - x_{i+1} = p_i$. To one-loop level, we find

$$\begin{aligned} \ln W_6 = a \left\{ -\frac{1}{2\epsilon^2} \sum_{i=1}^6 (-x_{i,i+2}^2 \mu^2)^\epsilon + \frac{1}{2} \sum_{i=1}^6 \left[-\ln \left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln \left(\frac{x_{i+1,i+3}^2}{x_{i,i+3}^2} \right) \right. \right. \\ \left. \left. + \frac{1}{4} \ln^2 \left(\frac{x_{i,i+3}^2}{x_{i+1,i+4}^2} \right) - \frac{1}{2} \text{Li}_2 \left(1 - \frac{x_{i,i+2}^2 x_{i+3,i+5}^2}{x_{i,i+3}^2 x_{i+2,i+5}^2} \right) \right] \right\} + O(a^2) \end{aligned} \quad (5.16)$$

¹⁶Strictly speaking the duality between $\ln W_6$ and $\ln M_6^{(\text{MHV})}$ holds up to unessential additive constant and involves a nontrivial redefinition of the infrared/UV regulators. We refer interested reader to [13] for more details.

where $a = g^2 N / (8\pi^2)$ and the periodicity condition $i \equiv i + 6$ is imposed. Combining the relations (5.11), (5.15) and (5.3), we obtain

$$\begin{aligned} \mathcal{A}_6^{\text{NMHV}} / W_6 &= (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^6 p_i \right) \frac{\langle 61 \rangle^4 \langle 45 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle 61 \rangle} \tilde{c}_{146} (1 + a V_{146}) \\ &\times \delta^{(8)} \left(\sum_{j=1}^6 \lambda_j^\alpha \eta_j^A \right) \delta^{(4)} \left(\eta_4 [56] + \eta_5 [64] + \eta_6 [45] \right) + (\text{cyclic}). \end{aligned} \quad (5.17)$$

To extract the six-gluon scattering amplitude $A_6(i^- j^- k^-)$ from this relation, we make use of (5.1), expand the product of two delta functions on the right-hand side of (5.17) and identify the coefficient in front of $\eta_i^4 \eta_j^4 \eta_k^4$. To simplify the calculation we choose the amplitude $A_6(4^- 5^- 6^-) \equiv A^{+++---}$ and concentrate on the terms $\sim \eta_4^4 \eta_5^4 \eta_6^4$ only. Making use of the identity

$$\delta^{(8)} \left(\sum_{j=1}^6 \lambda_j^\alpha \eta_j^A \right) = \langle ik \rangle^4 \delta^{(4)} \left(\eta_i + \sum_{j \neq k} \eta_j \frac{\langle jk \rangle}{\langle ik \rangle} \right) \delta^{(4)} \left(\eta_k + \sum_{j \neq i} \eta_j \frac{\langle ji \rangle}{\langle ki \rangle} \right) \quad (5.18)$$

and choosing appropriately the indices i and k , we find after some algebra

$$\begin{aligned} A^{+++---} / W_6 &= \tilde{C}(a) + \left(\frac{[23] \langle 56 \rangle}{x_{25}^2} \right)^4 \mathbb{P}^{-2} \tilde{C}(a) + \left(\frac{\langle 4 | x_{41} | 1 \rangle}{x_{25}^2} \right)^4 \mathbb{P} \tilde{C}(a), \\ &+ \left(\frac{[12] \langle 45 \rangle}{x_{36}^2} \right)^4 \mathbb{P}^2 \tilde{C}(a) + \left(\frac{\langle 6 | x_{63} | 3 \rangle}{x_{36}^2} \right)^4 \mathbb{P}^{-1} \tilde{C}(a), \end{aligned} \quad (5.19)$$

where the following notation was introduced

$$\tilde{C}(a) = \tilde{c}_{146} \frac{(\langle 61 \rangle \langle 45 \rangle x_{14}^2)^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle 61 \rangle} [1 + a V_{146}] \quad (5.20)$$

and \mathbb{P} generates the cyclic shift of indices $i \rightarrow i + 1$ so that $\mathbb{P}^k \tilde{C}(a)$ means that all indices in the expression for $\tilde{C}(a)$ should be shifted by $i \rightarrow i + k$.

5.3 Six-gluon NMHV amplitudes to one loop

To one-loop order, the six-gluon NMHV color-ordered amplitudes A^{+++---} , A^{++-+-} and A^{+-+--} were computed in [19]. To perform a comparison with (5.19) we will only need the expression for the first amplitude. It reads

$$A^{+++---} = A_{6;0} + g^2 A_{6;1} + O(g^4), \quad (5.21)$$

where the expansion coefficients are given by

$$\begin{aligned} A_{6;0} &= \frac{1}{2} [B_1 + B_2 + B_3] \\ A_{6;1} &= c_\Gamma N \left[B_1 F_6^{(1)} + B_2 F_6^{(2)} + B_3 F_6^{(3)} \right]. \end{aligned} \quad (5.22)$$

Here $c_\Gamma = (4\pi)^{-2+\epsilon}\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon)$ is a normalization factor and $F_6^{(i)}$ stand for a combination of box integrals evaluated within the dimensional regularization with $D = 4 - 2\epsilon$

$$\begin{aligned}
F_6^{(i)} = & -\frac{1}{2\epsilon^2} \sum_{k=1}^6 \left(\frac{\mu^2}{-x_{k,k+2}^2} \right)^\epsilon - \ln \left(\frac{x_{i,i+3}^2}{x_{i,i+2}^2} \right) \ln \left(\frac{x_{i,i+3}^2}{x_{i+1,i+3}^2} \right) \\
& - \ln \left(\frac{x_{i,i+3}^2}{x_{i+3,i+5}^2} \right) \ln \left(\frac{x_{i,i+3}^2}{x_{i+4,i+6}^2} \right) + \ln \left(\frac{x_{i,i+3}^2}{x_{i+2,i+4}^2} \right) \ln \left(\frac{x_{i,i+3}^2}{x_{i+5,i+1}^2} \right) \\
& + \frac{1}{2} \ln \left(\frac{x_{i,i+2}^2}{x_{i+3,i+5}^2} \right) \ln \left(\frac{x_{i+1,i+3}^2}{x_{i+4,i+6}^2} \right) + \frac{1}{2} \ln \left(\frac{x_{i-1,i+1}^2}{x_{i,i+2}^2} \right) \ln \left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2} \right) \\
& + \frac{1}{2} \ln \left(\frac{x_{i+2,i+4}^2}{x_{i+3,i+5}^2} \right) \ln \left(\frac{x_{i+4,i+6}^2}{x_{i+5,i+1}^2} \right) + \frac{\pi^2}{3}, \tag{5.23}
\end{aligned}$$

where $x_{i,i+3}^2$ and $x_{i,i+2}^2$ are Mandelstam variables expressed in terms of the dual coordinates. The explicit expressions for the functions B_a are [19]¹⁷

$$\begin{aligned}
B_1 = & i \frac{(x_{14}^2)^3}{\langle 12 \rangle \langle 23 \rangle [45] [56] \langle 1 | x_{14} | 4 \rangle \langle 3 | x_{36} | 6 \rangle} \tag{5.24} \\
B_2 = & \left(\frac{[23] \langle 56 \rangle}{x_{25}^2} \right)^4 \mathbb{P}^{-2} B_1 + \left(\frac{\langle 4 | x_{41} | 1 \rangle}{x_{25}^2} \right)^4 \mathbb{P} B_1, \\
B_3 = & \left(\frac{[12] \langle 45 \rangle}{x_{36}^2} \right)^4 \mathbb{P}^2 B_1 + \left(\frac{\langle 6 | x_{63} | 3 \rangle}{x_{36}^2} \right)^4 \mathbb{P}^{-1} B_1
\end{aligned}$$

We observe a remarkable similarity between these expressions and those in the expansion of the superamplitude (5.19).

To separate the infrared divergent and finite parts in A^{+++---} , we divide both sides of (5.21) by the Wilson loop W_6 given by (5.16)

$$A^{+++---}/W_6 = \frac{1}{2} B_1 \left(1 + a V_6^{(1)} \right) + \frac{1}{2} B_2 \left(1 + a V_6^{(2)} \right) + \frac{1}{2} B_3 \left(1 + a V_6^{(3)} \right) + O(\epsilon). \tag{5.25}$$

As was already mentioned, infrared divergences have a universal form for all scattering amplitudes and, therefore, given the fact that the Wilson loop W_6 captures the divergences of the MHV amplitudes, we expect that A^{+++---}/W_6 should be finite as $\epsilon \rightarrow 0$. Indeed, performing the calculation of A^{+++---}/W_6 to one loop, we find that the functions $V_6^{(i)}$ (with $i = 1, 2, 3$) do not contain infrared divergences and have the following form

$$V_6^{(i)} = -\ln u_i \ln u_{i+1} + \frac{1}{2} \sum_{k=1}^3 \left[\ln u_k \ln u_{k+1} + \text{Li}_2(1 - u_k) \right] \tag{5.26}$$

where the periodicity condition $u_{i+3} = u_i$ is implied and u_1, u_2 and u_3 are conformal cross-ratios in the dual coordinates

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}. \tag{5.27}$$

¹⁷In these relations we used the expressions for B_a from [19] in which we substituted the gluon momenta by the dual coordinates $p_i = x_i - x_{i+1}$ and used the notations for contraction of spinor indices specified in Appendix A.

Note that $V_6^{(i)}$ are invariant under cyclic shifts of the indices

$$\mathbb{P}^3 V_6^{(i)} = V_6^{(i+3)} = V_6^{(i)} \quad (5.28)$$

and, most importantly, $V_6^{(i)}$ are invariant under conformal transformations of x_i .

We observe here the same phenomenon as was already mentioned in Sect. 5.2. Namely, A^{+++---} and W_6 both contain divergences and their (dual) conformal invariance is broken in dimensional regularization. Nevertheless, the ratio A^{+++---}/W_6 is finite for $\epsilon \rightarrow 0$ and, as a result, it is invariant under conformal transformations in dual x -variables.

5.4 Conformal Ward identities for $n = 6$ NMHV superamplitude

Let us compare the two expressions for the amplitude A^{+++---} given by (5.19) and (5.21). We recall that the latter expression is the result of the perturbative one-loop calculation of Ref. [19] while the former expression comes from the expansion of $n = 6$ NMHV superamplitude and involves yet unknown function $\tilde{C}(a)$ defined in (5.20). This function depends in turn on the function \tilde{c}_{146} carrying the dependence on helicities of the external gluons and the scalar function V_{146} .

Matching (5.19) and (5.21) we take into account the identity (5.28) and find that the two expressions for the scattering amplitude coincide upon identification $\tilde{C}(a) = B_1(1 + aV_6^{(1)})/2$, or equivalently

$$\tilde{C}(a) = \frac{1}{2} B_1 \frac{\langle 12 \rangle \langle 23 \rangle \dots \langle 61 \rangle}{(x_{14}^2 \langle 61 \rangle \langle 45 \rangle)^4} \left(1 + aV_6^{(1)} \right) + O(a^2). \quad (5.29)$$

Then, we replace B_1 by its expression (5.24), compare the result with (5.20) and identify the expressions for the three-level helicity function \tilde{c}_{146}

$$\tilde{c}_{146} = \frac{1}{2} \frac{\langle 34 \rangle \langle 56 \rangle}{x_{14}^2 \langle 1 | x_{14} | 4 \rangle \langle 3 | x_{36} | 6 \rangle (\langle 45 \rangle \langle 61 \rangle)^3 [45][56]}, \quad (5.30)$$

and for the one-loop scalar function $V_{146} = V_6^{(1)}$

$$V_{146} = -\ln u_1 \ln u_2 + \frac{1}{2} \sum_{k=1}^3 \left[\ln u_k \ln u_{k+1} + \text{Li}_2(1 - u_k) \right]. \quad (5.31)$$

Substituting (5.30) and (5.31) into (5.11) and (5.13) we obtain the one-loop expression for the $n = 6$ NMHV superamplitude and the corresponding ratio function, respectively.

By the construction, the superamplitude $\mathcal{A}_6^{\text{MHV}}$ defined in (5.17) reproduces the known one-loop expression for a particular six-gluon NMHV amplitude A^{+++---} . At the same time, it also generate the two other gluon nMHV amplitudes A^{++-+-} and A^{+-+--} as well as many other amplitudes containing gluinos and scalars. In particular, it follows from our analysis that *all* tree-level NMHV amplitudes are described by the following compact expression

$$\begin{aligned} \mathcal{A}_{6;0}^{\text{MHV}} &= i(2\pi)^4 \frac{\delta^{(4)}(\sum_{i=1}^6 \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_{j=1}^6 \lambda_j^\alpha \eta_j^A)}{\langle 12 \rangle \langle 23 \rangle \dots \langle 61 \rangle} \\ &\times [\tilde{c}_{146} \langle 61 \rangle^4 \langle 45 \rangle^4 \delta^{(4)}(\eta_4[56] + \eta_5[64] + \eta_6[45]) + (\text{cyclic})] \end{aligned} \quad (5.32)$$

with \tilde{c}_{146} given by (5.30). Moreover, adding the one-loop perturbative correction to the ratio function (5.13) simply amounts to inserting the additional factor $(1 + aV_{146})$ involving (5.31)

$$R_6^{\text{NMHV}} = \tilde{c}_{146} \langle 61 \rangle^4 \langle 45 \rangle^4 [1 + aV_{146}] \delta^{(4)}(\eta_4[56] + \eta_5[64] + \eta_6[45]) + (\text{cyclic}) \quad (5.33)$$

As a nontrivial test of these relations, we have verified that, when expanded in powers of η 's, the expressions for $\mathcal{A}_{6;0}^{\text{MHV}}$ and $\mathcal{A}_{6;1}^{\text{MHV}}$ correctly reproduce all known expressions for tree-level and one-loop $n = 6$ NMHV scattering amplitudes.

Next we would like to check the transformation properties of the $n = 6$ NMHV superamplitude (5.11) under dual superconformal transformations. To this end, we examine the ratio function (5.13). As before, the only nontrivial transformations are conformal inversions. Since V_{146} is conformal invariant, Eq. (5.31), while Ξ_{146} transforms covariantly (see Eq. (6.7) below), we have to examine the action of conformal inversions on the function \tilde{c}_{146} , Eq. (5.30). To do this, it is convenient to obtain another, equivalent representation for \tilde{c}_{146} .

We recall that \tilde{c}_{146} is given by a linear combination of three terms (5.8) involving the functions c_{146} , c_{514} and c_{351} . Since the six-gluon NMHV superamplitude only depends on \tilde{c}_{146} , the definition of the latter functions is ambiguous. We can make use of this ambiguity to choose the following ansatz for c_{pqr} (with $1 \leq p, q, r \leq 6$ and $p \neq q \neq r$)

$$c_{pqr} = - \frac{\langle q-1 \, q \rangle \langle r-1 \, r \rangle}{x_{qr}^2 \langle p | x_{pr} x_{r \, q-1} | q-1 \rangle \langle p | x_{pr} x_{r \, q} | q \rangle \langle p | x_{pq} x_{q \, r-1} | r-1 \rangle \langle p | x_{pq} x_{q \, r} | r \rangle}. \quad (5.34)$$

where we used the notation for the contraction of spinor indices explained in Appendix A. It is straightforward to verify that the three terms inside the square brackets on the right-hand side of (5.8) produce the same contribution and reproduce the relation (5.30)

$$\tilde{c}_{146} = \frac{1}{2} c_{146}. \quad (5.35)$$

The relation (5.34) admits a natural generalization from $n = 6$ to arbitrary $n > 6$. As we will argue in the next section, the coefficient functions c_{pqr} enter the expression for the one-loop n -particle NMHV superamplitudes.

Another remarkable feature of (5.34) is that c_{pqr} is built from exactly those conformal covariant combinations of spinors that we already encountered before in Sect. 2.6. Making use of the relations (2.42) and (2.33), it is straightforward to verify that

$$I[c_{pqr}] = c_{pqr} (x_1^2 x_4^2 x_6^2)^4. \quad (5.36)$$

It is remarkable that c_{pqr} transforms covariantly under conformal inversions. Most importantly, the corresponding conformal weight is exactly the one that is needed to compensate the conformal weight of $\delta^{(4)}(\Xi_{146})$, Eq. (6.7). This means that the product $c_{146} \delta^{(4)}(\Xi_{146})$ is invariant under inversion and therefore under the $SO(2, 4)$ dual conformal transformations. One can verify that it is also invariant under superconformal transformations.

Thus, in agreement with our expectations, the ratio function (5.13) is superconformal invariant and verifies the conformal Ward identities (5.14). This does not imply however that the amplitude $\mathcal{A}_6^{\text{NMHV}}$ is covariant under these transformations. On the contrary, its conformal properties are broken by infrared divergences already at one loop but the corresponding anomalous contribution cancels against a similar contribution from $\mathcal{A}_6^{\text{MHV}}$ in such a way that the corresponding ratio function remains conformal.

6 Next-to-MHV superamplitude

In this section we propose the general form of the n -particle NMHV one-loop superamplitude. We construct it from a particular set of three-point nilpotent dual superconformal invariants, which encode the super-helicity structures. Each such invariant is accompanied by a finite and exactly dual conformal invariant function made of one-loop momentum integrals. We briefly discuss the twistor coplanarity properties of the super-helicity structures. We also propose a new, manifestly Lorentz covariant form of the n -particle NMHV tree-level superamplitude.

6.1 General structure of the NMHV superamplitude

The NMHV gluon amplitudes are characterized by the presence of three negative-helicity gluons. To describe them we need the second term in the expansion (3.19), i.e. a superamplitude of Grassmann degree of homogeneity 12. It contains terms of the type $(\eta_i)^4(\eta_j)^4(\eta_k)^4$, whose coefficients are the amplitudes with gluons of helicity -1 at points i, j, k and $+1$ elsewhere. By the same counting, in the MHV case the required Grassmann degree is 8, and it is indeed provided by the Grassmann delta function in (2.6).

We can assume that the entire MHV amplitude (3.22) should appear as a prefactor in the NMHV amplitude. The two delta functions are needed for conservation of the (super)momenta. Further, the bosonic denominator in (3.22) supplies the necessary helicity and conformal weights $+1$ at each point. Therefore, the generalization of the MHV superamplitude we are looking for should have the form

$$\mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \times [R_n^{\text{NMHV}} + O(\epsilon)] \quad (6.1)$$

where $R_n^{\text{NMHV}} \propto (\eta)^4$ is a new factor of Grassmann degree 4. Its perturbative expansion starts with a tree-level part, after which come the loop corrections. Since the MHV prefactor carries the necessary helicity and conformal weights, we deduce that R_n^{NMHV} must be a Lorentz scalar of vanishing helicity and be a *dual superconformal invariant*¹⁸.

Let us try to determine the number of such superconformal invariants. First of all, invariance under Q -supersymmetry (4.6) implies that they depend only on the Q -invariant variables η . Further, using the two projections of the Grassmann parameter $\bar{\rho}_{\dot{\alpha}}$ of \bar{S} -supersymmetry (4.23), we can set to zero any two η 's. Then, we can solve for another pair of η 's from the conservation condition (4.2). Thus, in the end we find only $n-4$ independent Q - and \bar{S} -supersymmetry invariant variables. We could use them as a basis for constructing $SU(4)$ invariants $\epsilon_{ABCD}\eta_k^A\eta_l^B\eta_m^C\eta_n^D$ of the required degree four. The last step would be to try to make all this dual conformal invariant (not forgetting that we need helicity weight zero). This is not so easy, given the inhomogeneous transformation law (4.22).

At first sight, the above procedure looks quite complicated. Fortunately, the relevant set of dual superconformal invariants is suggested to us by comparison with the results on the n -gluon NMHV amplitude of Bern et al [21].¹⁹ In the rest of this section we describe this construction and formulate our proposal for the complete n -particle NMHV superamplitude. This proposal generalizes the detailed analysis of the simplest, $n=6$ NMHV superamplitude in Sect. 5.

¹⁸At least under the action of the dual supersymmetry generators Q and \bar{S} , see the discussion in Sect. 6.5.

¹⁹These invariants can be directly obtained by a computation [41] using a supersymmetrized version of the generalized four-particle cut technique of [22].

6.2 Three-point superconformal covariants

Our main building block for the new Grassmann factor R_n^{NMHV} will be a set of dual superconformal invariants R_{pqr} which we construct in Sect. 6.3. These invariants are in turn made of dual superconformal covariants Ξ_{pqr}^A , which are linear combinations of θ 's labeled by a triplet of points p, q, r in dual superspace. They have the following manifest properties: (i) invariance under Q - and \bar{S} -supersymmetry; (ii) invariance under translations (P); (iii) covariance under inversion (I) and $SU(4)$. In addition, we want Ξ_{pqr}^A to be Lorentz scalars (but they will carry helicity at point p). The key idea in constructing such dual supercovariants is to use the two linear transformations with odd parameters (4.6), (4.15) in order to fix a coordinate frame in which two θ 's are set to zero. Such a choice is consistent with conformal invariance, as follows from (4.14). Then we can easily construct a conformal covariant from each of the remaining θ 's, which are Q - and \bar{S} -invariant in this frame. Finally, we can undo the supersymmetry transformation which lead to this frame, thus obtaining the manifest supercovariant Ξ_{pqr}^A .

Let us see how this works in detail.²⁰ Choose any triplet of distinct points p, q, r , $p \neq q \neq r$. Using (4.6), (4.15) we can shift away two of the θ 's of the triplet, e.g.,

$$\theta_q = \theta_r = 0 \quad (6.2)$$

To this end we have to find parameters $\epsilon, \bar{\rho}$ such that

$$\begin{aligned} \theta'_q &= \theta_q + \epsilon + x_q \bar{\rho} = 0 \\ \theta'_r &= \theta_r + \epsilon + x_r \bar{\rho} = 0 \end{aligned} \quad (6.3)$$

The solution of these linear equations is

$$\bar{\rho} = -x_{qr}^{-1}(\theta_q - \theta_r), \quad \epsilon = -x_q x_{qr}^{-1} \theta_r + x_r x_{qr}^{-1} \theta_q \quad (6.4)$$

where we are assuming that $x_{qr}^2 \neq 0$, i.e. $|q - r| \geq 2$.²¹

We remark that the case $|q - r| = 2$ is exceptional. Take, for instance, $q = r + 2$. Shifting away θ_r and θ_{r+2} implies $\theta_{r+1} = -\lambda_r \eta_r = \lambda_{r+1} \eta_{r+1}$. Then the linear independence of λ_r and λ_{r+1} yields $\theta_{r+1} = 0$, as well as $\eta_r = \eta_{r+1} = 0$. So, in this case we can shift away not two, but three neighboring θ 's.

In the special frame (6.2), an obvious (but certainly not unique) conformal covariant and Lorentz invariant, made of the remaining θ_p , is

$$\lambda_p^\alpha \theta_{p\alpha} \equiv \langle p | \theta_p \rangle \quad (6.5)$$

Let us now construct the supercovariant by ‘undoing’ the supersymmetry transformations (6.3) which lead to (6.2). This means to do another transformation with the same parameters. The result is (to put the covariant in a more symmetric form, we multiply it by x_{qr}^2)

$$\Xi_{pqr} = \Xi_{prq} = \langle p | x_{pq} x_{qr} | \theta_r \rangle + \langle p | x_{pr} x_{rq} | \theta_q \rangle + x_{qr}^2 \langle p | \theta_p \rangle \quad (6.6)$$

Note that this covariant depends on *three points* in dual superspace, $x_{p,q,r}$, $\theta_{p,q,r}$, as well as on the spinor variable λ_p . The latter carries helicity weight $-1/2$ at point p .

²⁰To simplify the notation, we suppress the $SU(4)$ index A .

²¹Recall that for two adjacent points in dual space, e.g., q and $q + 1$, the ‘distance’ $x_{qq+1}^2 = 0$.

This procedure automatically produces a dual conformal covariant. Indeed, using the transformation rules (2.24), (2.31) and (4.14), the covariance under inversion is manifest,

$$I[\Xi_{pqr}] = \frac{\Xi_{pqr}}{x_p^2 x_q^2 x_r^2} \quad (6.7)$$

The invariance under Q follows from the identity $x_{pq}x_{qr} + x_{pr}x_{rq} + x_{qr}^2 \mathbb{I} = 0$, which also allows us to rewrite (6.6) as follows:

$$\Xi_{pqr} = \langle p | [x_{pq}x_{qr}(|\theta_r\rangle - |\theta_p\rangle) + x_{pr}x_{rq}(|\theta_q\rangle - |\theta_p\rangle)] \quad (6.8)$$

$$= -\langle p | \left[x_{pq}x_{qr} \sum_{i=p+1}^{r-1} |i\rangle \eta_i + x_{pr}x_{rq} \sum_{i=p+1}^{q-1} |i\rangle \eta_i \right] \quad (6.9)$$

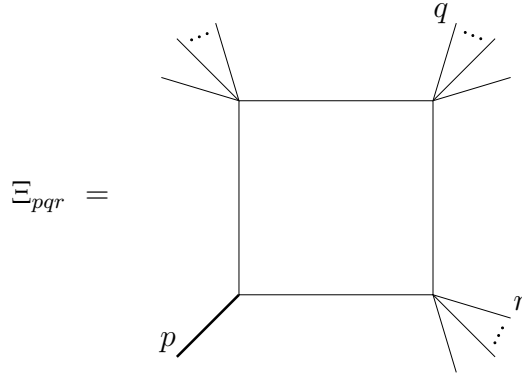
(where we have assumed that $p < \min(q, r)$). As explained earlier, these two symmetries yield invariance under \bar{S} (which is also very easy to check explicitly using (4.15)).

Note that besides the forbidden choice $|q - r| = 1$ (it is easy to see that (6.6) vanishes in this case), there is another case where the covariant (6.6) is trivial, $q = p + 1$ (or, equivalently, $r = p + 1$). The reason for this is simple. In the frame (6.3), if for example $q = p + 1$, we have $\theta_{p+1} = 0$. But then $\theta_p = \theta_{p+1} + |p\rangle\eta_p = |p\rangle\eta_p$, so $\langle p|\theta_p\rangle = 0$ and $\Xi_{p,p+1,r} = 0$.

Without loss of generality we can order the the points p, q, r clockwise on the circle, $p < q < r \pmod{n}$. Then the supercovariants (6.6) exist for the following choices of the indices:

$$\begin{aligned} 1 &\leq p \leq n \\ p+2 &\leq q \leq p-3 \pmod{n} \\ q+2 &\leq r \leq p-1 \pmod{n} \end{aligned} \quad (6.10)$$

A convenient way to represent such supercovariants pictorially is to use the box diagrams from [21]. There they were introduced to depict the coefficients of the three-mass box integrals in the one-loop NMHV amplitude.



It is easy to count the total number of supercovariants (6.6), $n(n-3)(n-4)/2$. Clearly, it largely exceeds the number $n-4$ of independent covariants that we found above. This suggests that there exist many identities relating various Ξ 's. Here we present only one of them, which was needed in Sect. 5. Consider the supercovariants $\Xi_{p,p+2,r}$ and $\Xi_{p+2,r,p+1}$ (the restriction $r \geq p+4$ is assumed; the points are ordered clockwise on the circle). Let us fix the frame $\theta_{p+2} = \theta_r = 0$. Using (6.8) and the fact that in this frame $\theta_{p+1} = |p+1\rangle\eta_{p+1}$, we can bring the two covariants to the form

$$\begin{aligned} \Xi_{p,p+2,r} &= x_{p+2,r}^2 \langle p|\theta_p\rangle = x_{p+2,r}^2 \langle p|\theta_{p+1}\rangle = x_{p+2,r}^2 \langle p|p+1\rangle\eta_{p+1} \\ \Xi_{p+2,r,p+1} &= \langle p+2|x_{p+2,r}x_{r,p+1}|\theta_{p+1}\rangle = -x_{p+2,r}^2 \langle p|p+1\rangle\eta_{p+1} \end{aligned} \quad (6.11)$$

hence the relation

$$\Xi_{p,p+2,r} = -\frac{\langle p|p+1\rangle}{\langle p+2|p+1\rangle} \Xi_{p+2,r,p+1} \quad (6.12)$$

6.3 Three-point superconformal invariants

Our next task is to construct the nilpotent factor R_n^{NMHV} (6.1) out of the superconformal covariants (6.6). To start with, we need to multiply four supercovariants (not necessarily identical) together, in order to form an $SU(4)$ invariant of the required odd degree four, $\epsilon_{ABCD} \Xi^A \Xi^B \Xi^C \Xi^D$. Then, remembering that R_n^{NMHV} has to be a dual conformal invariant without helicity (all the weights are carried by the MHV prefactor in (6.1)), we will need to find an appropriate bosonic factor which compensates the conformal and helicity weights of the Ξ 's. In principle, there are many ways how to do this, but one of them is very special – we wish to preserve the three-point nature of the supercovariant. Thus, we need to take four copies of the same²² Ξ_{pqr} and form the nilpotent $SU(4)$ invariant of degree four

$$\delta^{(4)}(\Xi_{pqr}) \equiv \frac{1}{4!} \epsilon_{ABCD} \Xi_{pqr}^A \Xi_{pqr}^B \Xi_{pqr}^C \Xi_{pqr}^D \quad (6.13)$$

According to (6.7), it transforms with conformal weight -4 at each point,

$$I[\delta^{(4)}(\Xi_{pqr})] = \frac{\delta^{(4)}(\Xi_{pqr})}{x_p^8 x_q^8 x_r^8} \quad (6.14)$$

and has helicity weight -2 at point p .

Next, we need to find a bosonic factor c_{pqr} which compensates the above weights. Once again, we wish to restrict ourselves to using only the three points $x_{p,q,r}$. Three-point string-type conformal covariants and Lorentz invariants are shown in (2.42). In our case, the following four such strings are possible (recall (2.20)):

$$\begin{aligned} \langle p|x_{pr}x_{rq}|q-1\rangle &\equiv \langle p|x_{pr}x_{rq-1}|q-1\rangle \\ \langle p|x_{pr}x_{rq}|q\rangle & \\ \langle p|x_{pq}x_{qr-1}|r\rangle &\equiv \langle p|x_{pq}x_{qr-1}|r-1\rangle \\ \langle p|x_{pq}x_{qr}|r\rangle & \end{aligned} \quad (6.15)$$

Multiplying them together, we obtain the necessary helicity weight -2 at point p . Finally, we just need a couple of additional obvious factors, which adjust the conformal weights and cancel the helicity weights at points $q-1, q, r-1, r$. The result is

$$c_{pqr} = \frac{\langle q-1|q\rangle \langle r-1|r\rangle}{x_{qr}^2 \langle p|x_{pr}x_{rq}|q-1\rangle \langle p|x_{pr}x_{rq}|q\rangle \langle p|x_{pq}x_{qr}|r-1\rangle \langle p|x_{pq}x_{qr}|r\rangle} \quad (6.16)$$

Thus, we propose to build the nilpotent factor R_n^{NMHV} (6.1) as a linear combination of the following three-point superconformal invariants:

$$R_{pqr} = c_{pqr} \delta^{(4)}(\Xi_{pqr}) \quad (6.17)$$

²²Any other $\Xi_{p'q'r'}$ will bring in at least one new point.

By construction, it is manifestly invariant under conformal inversion, as well as Q and \bar{S} supersymmetry. Remarkably, (6.17) turns out to be invariant under \bar{Q} supersymmetry (4.11) as well (and consequently under $S = I\bar{Q}I$). The best way to see this is to use the frame fixing procedure described above. We want to show that

$$\bar{Q}_{\dot{\alpha}}^A R_{pqr} = 0 \quad (6.18)$$

We already know that $\bar{S}_A^{\dot{\alpha}} R_{pqr} = 0$, so we derive the compatibility conditions (see (B.3)) $(D - C) R_{pqr} = M_{\beta}^{\dot{\alpha}} R_{pqr} = R_B^A R_{pqr} = 0$. They are trivially satisfied since R_{pqr} has neither conformal nor helicity weight, and is a Lorentz scalar and $SU(4)$ singlet. The same argument shows that the left-hand side of Eq. (6.18) is annihilated by \bar{S} (and also by Q , since $\{Q, \bar{Q}\} R_{pqr} = P R_{pqr} = 0$). This allows us to apply the combined Q and \bar{S} transformation (6.3) which leads to the fixed supersymmetry frame (6.2). Further, the generator \bar{Q} (4.11) only sees the three points $x_{p,q,r}$ inside R_{pqr} (6.17), but its action on $x_{q,r}$ is trivial in this fixed frame. Moreover, in the same frame we have $\Xi_{pqr} = x_{qr}^2 \langle p | \theta_p \rangle$ (see (6.5)), so

$$\begin{aligned} \bar{Q}_{\dot{\alpha}}^A R_{pqr} &= \theta_p^{A\alpha} \frac{\partial}{\partial x_p^{\dot{\alpha}a}} \left(\frac{\langle q-1|q\rangle \langle r-1|r\rangle x_{qr}^6 \delta^{(4)}(\langle p | \theta_p \rangle)}{\langle p | x_{pr} x_{rq} | q-1 \rangle \langle p | x_{pr} x_{rq} | q \rangle \langle p | x_{pq} x_{qr} | r-1 \rangle \langle p | x_{pq} x_{qr} | r \rangle} \right) \\ &\propto \langle p | \theta_p \rangle \delta^{(4)}(\langle p | \theta_p \rangle) = 0 \end{aligned} \quad (6.19)$$

The details of this proof explain why we insisted on the three-point nature of the superinvariant R_{pqr} – it allowed us, after fixing the frame (6.2), to drastically simplify the structure of the invariant and of the generator \bar{Q} , and to profit from the odd delta function $\delta^{(4)}(\Xi_{pqr})$ which annihilates the variation.

Summarizing the above argument, the minimal set of requirements that R_{pqr} must be annihilated by the generators of dual Poincaré supersymmetry Q, \bar{Q}, P, M , as well as being invariant under dual conformal inversion and $SU(4)$, implies that it is a *dual superconformal invariant* of the full $SU(2, 2|4)$. Further, if we restrict ourselves to *three-point invariants*, then (6.17) is the only solution. We point out once again that the three-point nature of the invariants has to do with the 3-mass-box one-loop integrals in the gluon amplitude of [21]. For NNMHV amplitudes, where also 4-mass boxes appear, the situation may change and we may have to use four-point invariants. This question is still under investigation.

Another interesting property of the superinvariants (6.17) is their independence of $\tilde{\lambda}$. Indeed, both the odd covariants Ξ_{pqr} (6.6) and the coefficients c_{pqr} (6.16) are expressed in terms of the dual *chiral* superspace coordinates x, θ, λ . The dependence on $\tilde{\lambda}$ is only implicit, through the x_{ij} . This is another manifestation of the holomorphic nature of the $\mathcal{N} = 4$ superamplitudes.

Finally, we note that the identity (6.12) between supercovariants can be upgraded to an identity between the corresponding superinvariants,

$$R_{p,p+2,r} = R_{p+2,r,p+1} \quad (6.20)$$

It is most easily shown in a fixed frame of the type (6.2).

6.4 Twistor coplanarity of the superinvariants

One of the main observations of Witten [1] was that the gluon amplitudes, formulated in the on-shell space with coordinates $\lambda, \tilde{\lambda}$, exhibit some unexpected simple geometric properties after

performing a ‘twistor transform’ (a partial Fourier transform of the variables $\tilde{\lambda}$, but not of λ). One such property is ‘twistor coplanarity’. Formulated in terms of the original amplitude $\mathcal{A}(\lambda, \tilde{\lambda})$ (before the twistor transform), it takes the form of a second-order differential constraint:

$$\mathcal{K}_{mnst} A(\lambda, \tilde{\lambda}) = 0 \quad (6.21)$$

where the coplanarity operator is

$$\mathcal{K}_{mnst} = \langle mn \rangle \kappa_{st} + \text{permutations}, \quad \kappa_{st} = \partial_{\dot{\alpha}s} \epsilon^{\dot{\alpha}\beta} \partial_{\dot{\alpha}t}, \quad \partial_{\dot{\alpha}s} \equiv \frac{\partial}{\partial \tilde{\lambda}_s^{\dot{\alpha}}} \quad (6.22)$$

for any set of four points m, n, s, t . This property has been widely discussed in the literature. In particular, in [21] it was shown that the 3-mass-box coefficients are annihilated by the coplanarity operator.

Here we demonstrate that our dual superinvariants R_{pqr} (which contain, among others, the 3-mass-box coefficients) are coplanar in the above sense, as a direct corollary of (i) \bar{Q} supersymmetry and (ii) a much simpler, supersymmetric version of (6.21). To do this, we regard the superinvariants as functions of the on-shell superspace variables, $R_{pqr}^{\text{NMHV}}(\lambda, \tilde{\lambda}, \eta)$. Let us then define the ‘super-coplanarity’ operator

$$\Omega_{ABst} = \Omega_{BAst} = \partial_{As} \partial_{Bt} - \partial_{At} \partial_{Bs}, \quad \partial_{As} \equiv \frac{\partial}{\partial \eta_s^A} \quad (6.23)$$

Now, let us apply this operator to R_{pqr} , assuming that both points s, t fall within the range of η ’s in Ξ_{pqr} (6.9) (otherwise Ω_{ABst} annihilates R_{pqr} trivially). The derivatives in (6.23) free two of the $SU(4)$ indices of ϵ_{ABCD} , which contradicts the symmetry of Ω . Thus,

$$\Omega_{ABst} R_{pqr} = 0 \quad (6.24)$$

We have also seen that R_{pqr} is annihilated by the dual supersymmetry generator \bar{Q} . Then, (anti)commuting \bar{Q} (in the form (4.13)) twice with Ω , we find

$$\{\bar{Q}_\beta^D, [\bar{Q}_\alpha^C, \Omega_{ABst}]\} \simeq \epsilon_{\dot{\alpha}\beta} \delta_{(A}^C \delta_{B)}^D \kappa_{st} \quad (6.25)$$

We thus see that the coplanarity of the superinvariants,

$$\mathcal{K}_{mnst} R_{pqr} = 0 \quad (6.26)$$

is an immediate consequence of super-coplanarity (6.24) and of the dual \bar{Q} supersymmetry. The former of these properties is valid at tree as well as at loop level (it is an obvious property of the super-helicity structures), while the latter is sensitive to the x -dependence and becomes anomalous in the presence of infrared divergent loop corrections.

6.5 The complete NMHV n -point one-loop superamplitude. New form of the tree-level amplitude

We are now ready to formulate our proposal for the complete one-loop NMHV n -point superamplitude:

$$\mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \left[\sum_{p,q,r=1}^n R_{pqr} (1 + a V_{pqr}(x_{ij}) + O(\epsilon)) + O(a^2) \right] \quad (6.27)$$

Here the nilpotent superconformal invariants R_{pqr} (6.17) encode the helicity structures for all the component amplitudes. In particular, expanding (6.27) in powers of η and collecting the terms $(\eta_{m_1})^4(\eta_{m_2})^4(\eta_{m_3})^4$, we obtain the helicity structures of the gluon amplitudes. This straightforward procedure (illustrated in detail in Sect. 5 for the case $n = 6$) exactly reproduces the full list of 3-mass-box coefficients from [21].

Further, the factors $V_{pqr}(x_{ij})$ in (6.27) contain the finite one-loop corrections (the infrared divergent part is in the MHV prefactor in (6.27)). They are given by appropriate combinations of 3-, 2- and 1-mass-box one-loop momentum integrals. One of our main claims is that $V_{pqr}(x_{ij})$ are *dual conformal invariant* functions of the dual space coordinates. The explicit $n = 6$ example worked out in detail in Sect. 5 confirms this conjecture. Further evidence in favor of the proposal (6.27) will be given in [41].

We recall that the manifest symmetries of each term in (6.27) are: helicity covariance, Lorentz invariance, as well as invariance under part of the dual superconformal algebra, namely Q, \bar{S}, P . The rest of the dual $SU(2, 2|4)$ could be obtained if we add \bar{Q} (and hence K, D and S) to the list of generators annihilating the superamplitude. Above we showed that the nilpotent structures R_{pqr} do indeed have this extended symmetry. However, the loop corrections $V_{pqr}^{(1)}(x_{ij})$ involve non-trivial dependence on the dual space coordinates, which is seen by the generator \bar{Q} . Moreover, as mentioned in Sect. 2.8, the MHV prefactor in (6.1) contains the dual conformal anomalous part of the amplitude, which inevitably must break the \bar{Q} symmetry (since \bar{S} is a manifest symmetry of the one-loop amplitude).

Collecting the a^0 terms in the perturbative expansion (6.27), we obtain the following new form of the *tree-level NMHV amplitude*:

$$\mathcal{A}_{n;0}^{\text{NMHV}} = \frac{\delta^{(4)}(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_{j=1}^n \lambda_j \eta_j)}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n 1 \rangle} \sum_{p,q,r} R_{pqr} \quad (6.28)$$

Expanding (6.28) in η , we have checked that its gluon tree-level components coincide with those given in [21]. The non-MHV tree-level amplitudes have been widely studied in the literature. In particular, Cachazo, Svrcek and Witten [42] have proposed a method for constructing them by combining MHV vertices. The characteristic feature of this approach is that it involves a constant fixed spinor (‘reference spinor’) which breaks manifest Lorentz invariance.²³ In [44] Georgiou, Glover and Khoze adapted the CSW construction for NMHV amplitudes to superspace. Their form of the tree amplitude bears some similarity to ours (6.28), but the main difference again is the use of a reference spinor and the loss of Lorentz invariance (see Appendix C for details).

7 Conclusions and outlook

In this paper, we have argued that the dual conformal symmetry previously observed for MHV amplitudes is part of a dual superconformal symmetry that governs the form of *all* (NMHV, NNMHV, ...) scattering amplitudes in $\mathcal{N} = 4$ SYM.

It is important to realize that dual conformal symmetry is broken by the infrared regulator. Therefore, in order to be able to make predictions about scattering amplitudes, one has to control how this symmetry is broken. Insight into the mechanism of breakdown of dual conformal

²³In [42] it was argued that the sum of all MHV \times MHV diagrams effectively does not depend on the reference spinor. Later on Kosower [43] was able to rewrite the CSW NMHV tree amplitude in a form where the reference spinor manifestly drops out.

symmetry has come from the study of a particular type of light-like Wilson loops, which are expected to be dual to MHV gluon scattering amplitudes [6, 9, 10, 11, 12, 15, 13]. The Wilson loops satisfy anomalous conformal Ward identities derived in [11, 12], and so do the MHV amplitudes, through the conjectured duality. More precisely, the finite part of the amplitude satisfies the anomalous Ward identity (see Sect. 2.8)

$$K^\mu F_n^{\text{MHV}} = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^n \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\mu + O(\epsilon). \quad (7.1)$$

As we have shown in this paper, the dual conformal properties of non-MHV amplitudes can be made manifest as well. In order to do this, one introduces the Grassmann variables η_i to describe the $\mathcal{N} = 4$ supermultiplet, together with dual superspace coordinates

$$\lambda_{i\alpha} \eta_i^A = \theta_{i\alpha}^A - \theta_{i+1\alpha}^A, \quad (7.2)$$

in close analogy with the dual bosonic coordinates defined by $\lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}} = x_{i\alpha\dot{\alpha}} - x_{i+1\alpha\dot{\alpha}}$. We conjecture that *all* amplitudes satisfy the same anomalous dual conformal Ward identities as the MHV amplitudes. In other words, we choose to write the full n -point superamplitude as

$$\mathcal{A}_n = \mathcal{A}_n^{\text{MHV}} [R_n(x_i, \theta_i, \lambda_i) + O(\epsilon)], \quad (7.3)$$

where $R_n = 1 + R_n^{(1)} + R_n^{(2)} + \dots + R_n^{(n-4)}$ and $R_n^{(p)}$ contains the terms corresponding to the N^p MHV amplitudes. Then the infrared divergences and the conformal anomaly are contained in $\mathcal{A}_n^{\text{MHV}}$, and therefore the (infrared finite) function R_n satisfies the conformal Ward identity

$$K^\mu R_n(x_i, \theta_i, \lambda_i) = 0, \quad (7.4)$$

with the conformal boost operator K^μ defined appropriately on the superspace, see equation (4.34) for its explicit expression. In order to check our conjecture (7.4), we wrote various one-loop NMHV amplitudes that were available in the literature [19, 20, 21, 27] in the superspace form using η_i . We illustrated this by giving the explicit example of the $n = 6$ NMHV superamplitude, where $R_6^{(1)}$ could be written in a manifestly dual conformal form, so that it obviously vanishes under the action of K^μ .

An equivalent way of stating our conjecture is the following. We can write the MHV factor in (7.3) as the product of its tree-level $\mathcal{A}_{n;0}^{\text{MHV}}$ and loop-correction $M_n(x_i; \mu, \epsilon)$ parts and then combine everything except the tree-level MHV factor into a single scalar function \mathcal{M}_n ,

$$\mathcal{A}_n = \mathcal{A}_{n;0}^{\text{MHV}} \mathcal{M}_n(x_i, \theta_i, \lambda_i; \mu, \epsilon). \quad (7.5)$$

The function \mathcal{M}_n is simply the product $M_n(x_i) [R_n(x_i, \theta_i, \lambda_i) + O(\epsilon)]$. Then one can take the logarithm of \mathcal{M}_n ,

$$\ln \mathcal{M}_n = [\text{IR divergences}] + F_n + O(\epsilon). \quad (7.6)$$

Finally the conjecture (7.4) can be restated as the fact that F_n (which is nothing but $F_n^{\text{MHV}} + \ln R_n$) satisfies the anomalous Ward identity,

$$K^\mu F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^n \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\mu + O(\epsilon). \quad (7.7)$$

The reason why the dual conformal properties of the amplitudes become apparent only in superspace is that under conformal boosts different component amplitudes transform into each other, and it is only the superamplitude which has definite conformal properties. We pointed out that there is a sub-class of amplitudes, the split-helicity amplitudes, which are exceptional. They do not 'mix' with other amplitudes in the above sense and therefore are dual conformal on their own. This allows us to suggest a two-loop test of our conjecture: for example, the $\mathcal{A}_6^{\text{NMHV}}$ amplitude with helicity assignment $(+++--)$ should be dual conformal on its own.

One of the results of this paper is a compact form for all NMHV one-loop amplitudes for arbitrary external particles, namely

$$\mathcal{A}_n^{\text{NMHV}} = \mathcal{A}_n^{\text{MHV}} \left[\sum_{p,q,r=1}^n R_{pqr} (1 + a V_{pqr}(x_{ij}) + O(\epsilon)) + O(a^2) \right] \quad (7.8)$$

where the dual superinvariants R_{pqr} were defined in (6.17), and $V_{pqr}(x_{ij})$ is a scalar dual conformal function. Beyond $n = 6$ we obtained (7.8) from an explicit calculation using the method of generalized unitarity cuts [22]. Using this manifestly supersymmetric setup for the unitarity cuts, the form (7.8) of the NMHV amplitudes emerges naturally. We will present the details of these results in a forthcoming publication.

We stress that (7.8) also provides an explicit form of the tree amplitude,

$$\mathcal{A}_{n;0}^{\text{NMHV}} = \frac{\delta^{(4)}(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum_{j=1}^n \lambda_j \eta_j)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \sum_{p,q,r} R_{pqr} \quad (7.9)$$

Such tree amplitudes appear for example when one computes one-loop amplitudes with a high number of external legs using the (generalized) unitarity cut method. Indeed, using maximal cuts, loop amplitudes can be constructed from the tree amplitudes, and therefore it is important to have explicit expressions for the latter. A similar application may be the computation of amplitudes in $\mathcal{N} = 8$ SYM using the KLT relations [45]. Notice that our formula for the tree-level NMHV amplitudes does not depend on a reference spinor, as compared to [42, 44, 46]. It would be interesting to see how formulae obtained from recursion relations including a reference spinor are equivalent to our formula.

It is natural to ask whether our amplitudes have a geometrical interpretation in (super-) twistor space. For example, MHV tree-level amplitudes lie on lines in supertwistor space [1]. The twistor space properties of NMHV gluon amplitudes were studied in [20, 21]. We find it likely that our NMHV superamplitudes will allow for an interpretation in supertwistor space. This point is currently under investigation.

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Appendices

A Notations and conventions

In this paper we use the two-component spinor formalism.

The dotted and undotted spinor indices are raised and lowered as follows:

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}}; \quad (\text{A.1})$$

$$\psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta, \quad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}} \quad (\text{A.2})$$

where the antisymmetric ϵ symbols have the properties:

$$\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -\epsilon^{12} = -\epsilon^{\dot{1}\dot{2}} = 1, \quad \epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma, \quad \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}} \quad (\text{A.3})$$

The convention for the contraction of a pair of spinor indices is

$$\psi^\alpha \lambda_\alpha \equiv \langle \psi \lambda \rangle, \quad \bar{\chi}_{\dot{\alpha}} \bar{\rho}^{\dot{\alpha}} \equiv [\bar{\chi} \bar{\rho}] \quad (\text{A.4})$$

Two-component spinors satisfy the cyclic identity

$$\langle \psi \lambda \rangle \chi_\alpha + \langle \lambda \chi \rangle \psi_\alpha + \langle \chi \psi \rangle \lambda_\alpha = 0 \quad (\text{A.5})$$

which simply means that the antisymmetrization over three two-component indices vanishes identically.

The sigma matrices σ^μ are defined as follows:

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = (1, \vec{\sigma})_{\alpha\dot{\alpha}}, \quad (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\alpha\beta} (\sigma^\mu)_{\beta\dot{\beta}} = (1, -\vec{\sigma})^{\dot{\alpha}\alpha} \quad (\text{A.6})$$

and have the basic properties:

$$\begin{aligned} \sigma^\mu \tilde{\sigma}^\nu &= \eta^{\mu\nu} - i\sigma^{\mu\nu}, & \tilde{\sigma}^\mu \sigma^\nu &= \eta^{\mu\nu} - i\tilde{\sigma}^{\mu\nu}, \\ (\sigma^\mu)_{\alpha\dot{\alpha}} (\tilde{\sigma}_\mu)^{\dot{\beta}\beta} &= 2\delta_{\alpha\dot{\alpha}}^{\dot{\beta}\beta}, & (\sigma_\mu)_{\alpha\dot{\alpha}} (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} &= 2\delta_\mu^\alpha, \\ \sigma^{\mu\nu} &= -\sigma^{\nu\mu}, & \tilde{\sigma}^{\mu\nu} &= -\tilde{\sigma}^{\nu\mu}, & (\sigma^{\mu\nu})_\alpha{}^\alpha &= (\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}}{}^{\dot{\alpha}} = 0 \end{aligned} \quad (\text{A.7})$$

A four-vector x^μ can be written as a two-component bispinor:

$$x_{\alpha\dot{\alpha}} = x^\mu (\sigma_\mu)_{\alpha\dot{\alpha}}, \quad x^{\dot{\alpha}\alpha} = x^\mu \tilde{\sigma}_\mu^{\dot{\alpha}\alpha}, \quad x^\mu = \frac{1}{2} x^{\dot{\alpha}\alpha} \sigma_{\alpha\dot{\alpha}}^\mu \quad (\text{A.8})$$

Its square x^2 can be expressed in various ways:

$$x^2 = x^\mu x_\mu = \frac{1}{2} x^{\dot{\alpha}\alpha} x_{\alpha\dot{\alpha}}, \quad x_{\alpha\dot{\alpha}} x^{\dot{\alpha}\beta} = x^2 \delta_\alpha^\beta, \quad x^{\dot{\alpha}\alpha} x_{\alpha\dot{\beta}} = x^2 \delta_{\dot{\beta}}^{\dot{\alpha}} \quad (\text{A.9})$$

Its ‘inverse’ x^μ/x^2 takes the matrix form

$$(x^{-1})_{\alpha\dot{\alpha}} = \frac{x_{\alpha\dot{\alpha}}}{x^2}, \quad (x^{-1})_{\alpha\dot{\alpha}} x^{\dot{\alpha}\beta} = \delta_\alpha^\beta, \quad x^{\dot{\alpha}\alpha} (x^{-1})_{\alpha\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}} \quad (\text{A.10})$$

We often have to deal with ‘strings’ of commuting spinors $\lambda, \tilde{\lambda}$ and vectors x , for which we use the following short-hand notations, e.g.:

$$\begin{aligned} \langle p | x_{mn} x_{kl} | q \rangle &= \lambda_p^\alpha (x_{mn})_{\alpha\dot{\alpha}} (x_{kl})^{\dot{\alpha}\beta} \lambda_{q\beta} = -\langle q | x_{kl} x_{mn} | p \rangle \\ \langle p | x_{mn} | q \rangle &= \lambda_p^\alpha (x_{mn})_{\alpha\dot{\alpha}} \tilde{\lambda}_q^{\dot{\alpha}}, \quad \text{etc.} \end{aligned} \quad (\text{A.11})$$

B Conventional and dual superconformal generators

In this appendix we give the conventional and dual representations of the superconformal algebra. We begin by listing the commutation relations of the algebra $u(2, 2|4)$. The Lorentz generators $\mathbb{M}_{\alpha\beta}$, $\overline{\mathbb{M}}_{\dot{\alpha}\dot{\beta}}$ and the $su(4)$ generators \mathbb{R}^A_B act canonically on the remaining generators carrying Lorentz or $su(4)$ indices. The dilatation \mathbb{D} and hypercharge \mathbb{B} act via

$$[\mathbb{D}, \mathbb{J}] = \dim(\mathbb{J}), \quad [\mathbb{B}, \mathbb{J}] = \text{hyp}(\mathbb{J}). \quad (\text{B.1})$$

The non-zero dimensions and hypercharges of the various generators are

$$\begin{aligned} \dim(\mathbb{P}) &= 1, & \dim(\mathbb{Q}) &= \dim(\overline{\mathbb{Q}}) = \frac{1}{2}, & \dim(\mathbb{S}) &= \dim(\overline{\mathbb{S}}) = -\frac{1}{2} \\ \dim(\mathbb{K}) &= -1, & \text{hyp}(\mathbb{Q}) &= \text{hyp}(\overline{\mathbb{S}}) = \frac{1}{2}, & \text{hyp}(\overline{\mathbb{Q}}) &= \text{hyp}(\mathbb{S}) = -\frac{1}{2}. \end{aligned} \quad (\text{B.2})$$

The remaining non-trivial commutation relations are,

$$\begin{aligned} \{\mathbb{Q}_{\alpha A}, \overline{\mathbb{Q}}_{\dot{\alpha}}^B\} &= \delta_A^B \mathbb{P}_{\alpha\dot{\alpha}}, & \{\mathbb{S}_{\alpha A}, \overline{\mathbb{S}}_{\dot{\alpha}}^B\} &= \delta_A^B \mathbb{K}_{\alpha\dot{\alpha}}, \\ [\mathbb{P}_{\alpha\dot{\alpha}}, \mathbb{S}_{\beta}^A] &= \epsilon_{\dot{\alpha}\dot{\beta}} \overline{\mathbb{Q}}_{\dot{\alpha}}^A, & [\mathbb{K}_{\alpha\dot{\alpha}}, \mathbb{Q}_{\beta A}] &= \epsilon_{\alpha\beta} \overline{\mathbb{S}}_{\dot{\alpha}}^A, \\ [\mathbb{P}_{\alpha\dot{\alpha}}, \overline{\mathbb{S}}_{\dot{\beta} A}] &= \epsilon_{\dot{\alpha}\dot{\beta}} \mathbb{Q}_{\alpha A}, & [\mathbb{K}_{\alpha\dot{\alpha}}, \overline{\mathbb{Q}}_{\dot{\beta} A}] &= \epsilon_{\dot{\alpha}\dot{\beta}} \mathbb{S}_{\alpha A}, \\ [\mathbb{K}_{\alpha\dot{\alpha}}, \mathbb{P}^{\beta\dot{\beta}}] &= \delta_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}} \mathbb{D} + \mathbb{M}_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}} + \overline{\mathbb{M}}_{\dot{\alpha}}^{\dot{\beta}} \delta_{\alpha}^{\beta}, \\ \{\mathbb{Q}_{\alpha A}, \mathbb{S}_{\beta}^B\} &= \epsilon_{\alpha\beta} \mathbb{R}^B_A + \mathbb{M}_{\alpha\beta} \delta_A^B + \epsilon_{\alpha\beta} \delta_A^B (\mathbb{D} + \mathbb{C}), \\ \{\overline{\mathbb{Q}}_{\dot{\alpha}}^A, \overline{\mathbb{S}}_{\dot{\beta} B}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} \mathbb{R}^A_B + \overline{\mathbb{M}}_{\dot{\alpha}\dot{\beta}} \delta_B^A + \epsilon_{\dot{\alpha}\dot{\beta}} \delta_B^A (\mathbb{D} - \mathbb{C}). \end{aligned} \quad (\text{B.3})$$

We now give the generators in both the conventional and dual representations of the superconformal algebra. We will use the following shorthand notation:

$$\partial_{i\alpha\dot{\alpha}} = \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}, \quad \partial_{i\alpha A} = \frac{\partial}{\partial \theta_i^{\alpha A}}, \quad \partial_{i\alpha} = \frac{\partial}{\partial \lambda_i^{\alpha}}, \quad \partial_{i\dot{\alpha}} = \frac{\partial}{\partial \bar{\lambda}_i^{\dot{\alpha}}}, \quad \partial_{iA} = \frac{\partial}{\partial \eta_i^A}. \quad (\text{B.4})$$

We first give the generators of the conventional superconformal symmetry, using lower case characters to distinguish these generators from the dual superconformal generators which follow afterwards.

$$\begin{aligned} p^{\alpha\dot{\alpha}} &= \sum_i \lambda_i^{\alpha} \bar{\lambda}_i^{\dot{\alpha}}, & k_{\alpha\dot{\alpha}} &= \sum_i \partial_{i\alpha} \partial_{i\dot{\alpha}}, \\ \overline{m}_{\dot{\alpha}\dot{\beta}} &= \sum_i \bar{\lambda}_{i(\dot{\alpha}} \partial_{i\dot{\beta})}, & m_{\alpha\beta} &= \sum_i \lambda_{i(\alpha} \partial_{i\beta)}, \\ d &= \sum_i [\frac{1}{2} \lambda_i^{\alpha} \partial_{i\alpha} + \frac{1}{2} \bar{\lambda}_i^{\dot{\alpha}} \partial_{i\dot{\alpha}} + 2], & r^A_B &= \sum_i [\eta_i^A \partial_{iB} - \frac{1}{4} \eta_i^C \partial_{iC}], \\ q^{\alpha A} &= \sum_i \lambda_i^{\alpha} \eta_i^A, & \bar{q}_{\dot{A}}^{\dot{\alpha}} &= \sum_i \bar{\lambda}_i^{\dot{\alpha}} \partial_{i\dot{A}}, \\ s_A^{\alpha} &= \sum_i \partial_{i\alpha} \partial_{iA}, & \bar{s}_{\dot{A}}^{\dot{\alpha}} &= \sum_i \eta_i^{\dot{\alpha}} \partial_{i\dot{A}}. \end{aligned}$$

We can construct the generators of dual superconformal transformations by starting with the standard chiral representation and extending the generators so that they commute with the constraints,

$$(x_i - x_{i+1})_{\alpha\dot{\alpha}} - \lambda_{i\alpha} \tilde{\lambda}_{i\dot{\alpha}} = 0, \quad (\theta_i - \theta_{i+1})_{\alpha}^A - \lambda_{i\alpha} \eta_i^A = 0. \quad (\text{B.5})$$

By construction they preserve the surface defined by these constraints, which is where the amplitude has support. The generators are

$$P_{\alpha\dot{\alpha}} = \sum_i \partial_{i\alpha\dot{\alpha}}, \quad (\text{B.6})$$

$$Q_{\alpha A} = \sum_i \partial_{i\alpha A}, \quad (\text{B.7})$$

$$\bar{Q}_{\dot{\alpha}}^A = \sum_i [\theta_i^{\alpha A} \partial_{i\alpha\dot{\alpha}} + \eta_i^A \partial_{i\dot{\alpha}}], \quad (\text{B.8})$$

$$M_{\alpha\beta} = \sum_i [x_{i(\alpha} \dot{\alpha} \partial_{i\beta)\dot{\alpha}} + \theta_{i(\alpha}^A \partial_{i\beta)A} + \lambda_{i(\alpha} \partial_{i\beta)}], \quad (\text{B.9})$$

$$\bar{M}_{\dot{\alpha}\dot{\beta}} = \sum_i [x_{i(\dot{\alpha}}^{\alpha} \partial_{i\dot{\beta})\alpha} + \bar{\lambda}_{i(\dot{\alpha}} \partial_{i\dot{\beta})}], \quad (\text{B.10})$$

$$R^A_B = \sum_i [\theta_i^{\alpha A} \partial_{i\alpha B} + \eta_i^A \partial_{iB} - \frac{1}{4} \delta_B^A \theta_i^{\alpha C} \partial_{i\alpha C} - \frac{1}{4} \eta_i^C \partial_{iC}], \quad (\text{B.11})$$

$$D = \sum_i [x_i^{\alpha\dot{\alpha}} \partial_{i\alpha\dot{\alpha}} + \frac{1}{2} \theta_i^{\alpha A} \partial_{i\alpha A} + \frac{1}{2} \lambda_i^{\alpha} \partial_{i\alpha} + \frac{1}{2} \bar{\lambda}_i^{\dot{\alpha}} \partial_{i\dot{\alpha}}], \quad (\text{B.12})$$

$$C = \sum_i [\frac{1}{2} [\lambda_i^{\alpha} \partial_{i\alpha} - \bar{\lambda}_i^{\dot{\alpha}} \partial_{i\dot{\alpha}} - \eta_i^A \partial_{iA}], \quad (\text{B.13})$$

$$S_{\alpha}^A = \sum_i [\theta_{i\alpha}^B \theta_i^{\beta A} \partial_{i\beta B} - x_{i\alpha} \dot{\beta} \theta_i^{\beta A} \partial_{\beta\dot{\beta}} - \lambda_{i\alpha} \theta_i^{\gamma A} \partial_{i\gamma} - x_{i+1\alpha} \dot{\beta} \eta_i^A \partial_{i\dot{\beta}} + \theta_{i+1\alpha}^B \eta_i^A \partial_{iB}], \quad (\text{B.14})$$

$$\bar{S}_{\dot{\alpha}A} = \sum_i [x_{i\dot{\alpha}}^{\beta} \partial_{i\beta A} + \bar{\lambda}_{i\dot{\alpha}} \partial_{iA}], \quad (\text{B.15})$$

$$K_{\alpha\dot{\alpha}} = \sum_i [x_{i\alpha} \dot{\beta} x_{i\dot{\alpha}}^{\beta} \partial_{i\beta\dot{\beta}} + x_{i\dot{\alpha}}^{\beta} \theta_{i\alpha}^B \partial_{i\beta B} + x_{i\dot{\alpha}}^{\beta} \lambda_{i\alpha} \partial_{i\beta} + x_{i+1\alpha} \dot{\beta} \bar{\lambda}_{i\dot{\alpha}} \partial_{i\dot{\beta}} + \bar{\lambda}_{i\dot{\alpha}} \theta_{i+1\alpha}^B \partial_{iB}]. \quad (\text{B.16})$$

We also have the hypercharge B ,

$$B = \sum_i [\frac{1}{2} [\theta_i^{\alpha A} \partial_{i\alpha A} + \lambda_i^{\alpha} \partial_{i\alpha} - \bar{\lambda}_i^{\dot{\alpha}} \partial_{i\dot{\alpha}}]] \quad (\text{B.17})$$

Note that if we restrict the dual generators \bar{Q}, \bar{S} to the on-shell superspace they become identical to the conventional generators \bar{s}, \bar{q} .

C NMHV tree-level amplitudes with and without reference spinor

Let us rewrite the tree amplitude (6.28), in a way such that it resembles the GGK tree amplitude [44] with the reference spinor of CSW [42]. Define $\langle I_q | = \langle 1 | x_{1r} x_{qr}$ and $\langle I_r | = \langle 1 | x_{1q} x_{qr}$ (with

$\langle I_q | - \langle I_r | = x_{qr}^2 \langle 1 |$. Then (6.28) becomes (we have singled out the term $p = 1$)

$$\begin{aligned} \mathcal{A}_{n;0}^{\text{NMHV}} &= \delta^{(4)}\left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i\right) \delta^{(8)}\left(\sum_{j=1}^n \lambda_j \eta_j\right) \\ &\times \left[\sum_{q,r} \frac{\delta^{(4)}\left(\sum_{k=q}^{r-1} \langle I_r k \rangle \eta_k + \sum_{k=1}^{q-1} (\langle I_r k \rangle - \langle I_q k \rangle) \eta_k\right)}{x_{qr}^2 \langle 1 2 \rangle \dots \langle q-1 I_q \rangle \langle I_q q \rangle \dots \langle r-1 I_r \rangle \langle I_r r \rangle \dots \langle n 1 \rangle} \right. \\ &\quad \left. + \text{ cycle} \right] \end{aligned} \quad (\text{C.1})$$

Now, compare this to Eqs. (5.9), (5.10) from [44] for the same amplitude (with the identification $i \equiv q-1$, $j \equiv r-1$, $q_I^2 \equiv x_{qr}^2$; the argument of GGK's $\delta^{(4)}$ should be replaced by the complementary cluster n_2). GGK have a unique $\langle I | = [\xi_{\text{ref}} | x_{qr}$ defined with the help of CSW's reference spinor $[\xi_{\text{ref}} |$. Their amplitude has the form:

$$\begin{aligned} \mathcal{A}_{n;0}^{\text{CSW-GGK}} &= \delta^{(4)}\left(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i\right) \delta^{(8)}\left(\sum_{j=1}^n \lambda_j \eta_j\right) \\ &\times \sum_{q,r} \frac{\delta^{(4)}\left(\sum_{k=q}^{r-1} \langle I k \rangle \eta_k\right)}{x_{qr}^2 \langle 1 2 \rangle \dots \langle q-1 I \rangle \langle I q \rangle \dots \langle r-1 I \rangle \langle I r \rangle \dots \langle n 1 \rangle} \end{aligned} \quad (\text{C.2})$$

The main difference is that in (C.1) we are using two spinors, $[\xi_q | = \langle 1 | x_{1q}$ and $[\xi_r | = \langle 1 | x_{1r}$ (being expressed in terms of the external particle variables, they are not 'reference' spinors), while in (C.2) they have been merged into a single *constant* spinor, $[\xi_q | = [\xi_r | \equiv [\xi_{\text{ref}} |$, independent on the external particle variables.

It is an interesting question to find out how the two expressions (C.1) and (C.2) are equivalent.

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